A new type of difference class of interval numbers

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Abstract

In this article we introduce the notation difference operator $\Delta_m$ ($m \geq 0$ be an integer) for studying some properties defined with interval numbers. We introduced the classes of sequence $\bar{c}(p)(\Delta_m)$, $\bar{c}(p)(\Delta_m)$ and $\bar{c}_0(p)(\Delta_m)$ and investigate different algebraic properties like completeness, solidness, convergence free etc.

Key Words : Interval number, Completeness, Solid, Convergence free.

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1. Introduction

The concept of interval arithmetic was first suggested by Dwyer [15] in 1951. Thereafter the concept has been using in area of science and technology. The evidence of its development as a formal system and application in computational device is found in Moore [8], Moore and Yang [9] and others ([15], [16], [17] and [20]). Different mathematical concepts were introduced and studied with interval numbers by several researchers across the globe. Chiao [13] introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Sengnl and Eryilmaz [14] introduced and studied bounded and convergent sequence spaces of interval numbers and proved that these spaces are complete metric space. Recently Esi [1-8], Esi and Braha [18], Esi and Esi [19], Esi and Hazarika [20] and Esi and Catalbas [21] introduced and studied strongly almost-convergence and statistically almost-convergence of interval numbers.

A set consisting of a closed interval of real numbers $x$ such that $a \leq x \leq b$ is called an interval number. A real interval can also be considered as a set. We can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties. We denote the set of all real valued closed intervals by $R$. Any elements of $R$ is called closed interval and denoted by $\bar{x}$, that is $\bar{x} = \{x \in R : a \leq x \leq b\}$. An interval number $\bar{x}$ is a closed subset of real numbers [15]. Let $x_1$ and $x_r$ be first and last points of interval number $\bar{x}$, respectively then we have for $x_1, x_2 \in R$,

i) $\bar{x}_1 = \bar{x}_2 \Leftrightarrow x_1 = x_2, x_1 = x_2$,

ii) $\bar{x}_1 + \bar{x}_2 = \{x \in R : x_1 + x_2 \leq x \leq x_1 + x_2\}$

iii) $\alpha \bar{x} = \{x \in R : \alpha x_1 \leq x \leq \alpha x_1\}, \text{ for } \alpha \geq 0$ and $\alpha x = \{x \in R : \alpha x_1 \leq x \leq \alpha x_1\}, \text{ for } \alpha < 0$.

iv) $\bar{x}_1 \bar{x}_2 = \{x \in R : \min\{x_1, x_2\}, x_1, x_2, x_1, x_2, x_1, x_2\} \leq x$

\[ \leq \max\{x_1, x_2, x_1, x_2, x_1, x_2, x_1, x_2\} \]

The set of all interval numbers $R$ is a complete metric space defined by

\[ d(\bar{x}_1, \bar{x}_2) = \max\{|x_1 - x_2|, |x_1 - x_2|\} \]
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In the special case, $\bar{x}_1 = [a, a]$ and $\bar{x}_2 = [b, b]$, we obtain usual metric of $\mathbb{R}$.

Consider the transformation $f : N \to \mathbb{R}$, by $k \to f(k) = \bar{x}, x = (x_k)$, then $\bar{x} = (\bar{x}_k)$ is called sequence of interval numbers. The term $\bar{x}_k$ is called the $k$th term of sequence $(\bar{x}) = (\bar{x}_k)$.

By $\mathcal{W}$ we denotes the set of all interval numbers with real terms. We give the following definitions of convergence of interval numbers.

A sequence $\bar{x} = (\bar{x}_k)$ of interval numbers is said to be convergent to the interval number $\bar{x}_0$ if for each $\varepsilon > 0$ there exists a positive integer $k_0$ such that $d(\bar{x}_k, \bar{x}_0) < \varepsilon$ for all $k \geq k_0$, denoted by $\lim_{k\to \infty} \bar{x}_k = \bar{x}_0$. This imply that

$$\lim_{k\to \infty} \bar{x}_k = \bar{x}_0 \iff \lim_{k\to \infty} x_k = x_0 \quad \text{and} \quad \lim_{k\to \infty} x_k = x_0.$$  

An interval valued sequence space $\bar{E}$ is said to be solid if $\bar{y} = (\bar{y}_k) \in \bar{E}$ whenever $|\bar{y}_k| \leq |\bar{x}_k|$, for all $k \in \mathbb{N}$ and $\bar{x} = (\bar{x}_k) \in \bar{E}$.

An interval valued sequence space $\bar{E}$ is said to be monotone if $\bar{E}$ contains the canonical pre- image of all its step spaces.

An interval valued sequence space $\bar{E}$ is said to be convergence free if $\bar{y} = (\bar{y}_k) \in \bar{E}$ whenever $\bar{x} = (\bar{x}_k) \in \bar{E}$ and $\bar{x}_k = \bar{0}$ implies $\bar{y}_k = \bar{0}$.

Throughout the paper, $p = (p_k)$ is a sequence of bounded strictly positive numbers.

Esi[1] define the following interval valued sequence space:

$$\bar{c}(p) = \left\{ \bar{x} = (\bar{x}_k) : \sum_{k=1}^{\infty} [d(\bar{x}_k, \bar{0})]^{p_k} < \infty \right\},$$

for $p_k = 1$ for all $k \in \mathbb{N}$, we have

$$\bar{c}(p) = \left\{ \bar{x} = (\bar{x}_k) : \sum_{k=1}^{\infty} [d(\bar{x}_k, \bar{0})] < \infty \right\}.$$  

Kizmaz [12] defined the sequence space for crisp set. The concept further generalized by Tripathy and Esi [12] as follows:

Let $m > 0$ be an integer then $Z_1(\Delta_m) = \{(\bar{x}_k) \in w : (\Delta_m x_k) \in Z_1\}$, for $Z_1 = \ell_\infty, c$ and $c_0$. Where $\Delta_m x_k = x_k - x_{k+m}$, for all $k \in \mathbb{N}$ and they showed that these are Banach spaces under the norm $\|x\|_{\Delta_m} = \sum_{r=1}^{m} |x_r| + \sup_k |\Delta_m x_k|$. For $m = 1$, the sequence spaces $\ell_\infty(\Delta), c(\Delta)$ and $c_0(\Delta)$ are studied by Kizmaz [12].
In this paper we introduce the difference operator for sequence of interval numbers generalized by Tripathy and Esi [22] as follows:

Let \( \bar{x} = (\bar{x}_k) \) be a sequence of interval numbers and \( p = (p_k) \) is a sequence of bounded strictly positive numbers. Let \( m \geq 0 \) be an integer then

\[
Z(\Delta_m) = \{(\bar{x}_k) \in w^i : (\Delta_m \bar{x}_k) \in Z \}
\]

for \( Z = \ell(p)(\Delta_m) \) and \( c_0(p)(\Delta_m) \), where \( \Delta_m x_k = x_k - x_{k+m} \) for all \( k \in N \).

2. Main Results

**Theorem 2.1:** The sequence spaces \( \ell(p)(\Delta_m) \), \( \bar{c}(p)(\Delta_m) \) and \( c_0(p)(\Delta_m) \) are complete metric space with respect to the metric defined by

\[
\rho(\bar{x}, \bar{y}) = \sum_{k=1}^{\infty} [d(\bar{x}_k, \bar{y}_k)]^{p_k} + \sup_k [d(\Delta_m \bar{x}_k, \Delta_m \bar{y}_k)]
\]

**Proof:** Let \( (\bar{x}^i) \) be a Cauchy sequence in \( \ell(p)(\Delta_m) \) such that \( \bar{x}^i = (\bar{x}^i_k) = (\bar{x}^i_1, \bar{x}^i_2, \bar{x}^i_3, ...) \in \ell(p)(\Delta_m) \) for each \( i \in N \). Then for a given \( \varepsilon > 0 \), there exists \( n_0 \in N \), such that

\[
\rho(\bar{x}^i, \bar{x}^j) = \sum_{k=1}^{\infty} [d(\bar{x}^i_k, \bar{x}^j_k)]^{p_k} + \sup_k [d(\Delta_m \bar{x}^i_k, \Delta_m \bar{x}^j_k)] < \varepsilon, \text{ for all } i, j \geq n_0
\]

(2.1)

Then

\[
\sum_{k=1}^{\infty} [d(\bar{x}^i_k, \bar{x}^j_k)]^{p_k} < \varepsilon, \text{ for all } i, j \geq n_0
\]

\[
\Rightarrow d(\bar{x}^i_k, \bar{x}^j_k) < \varepsilon, \text{ for all } i, j \geq n_0 \text{ and for all } k \in N.
\]

\[
\Rightarrow (\bar{x}^i_k) \text{ is a Cauchy sequence in } \mathbb{R} \text{ and for all } k \in N.
\]

\[
\Rightarrow (\bar{x}^i_k) \text{ Converges in } \mathbb{R} \text{ and for all } k \in N \text{ as } \mathbb{R} \text{ is a Banach space.}
\]
Let \( \lim_{j} \bar{x}^j_k = \bar{x}_k \) (say) for each \( k \in N \) and \( \bar{x} = (\bar{x}_k) \).

From definition (2.1) we have
\[
d(\Delta_m \bar{x}^i_k, \bar{x}^j_k) < \varepsilon, \quad \text{for } i, j \geq n_0 \text{ and for } k \in N.
\]

\[\Rightarrow (\Delta_m \bar{x}^i_k) \text{ is a Cauchy sequence in } R \text{ for all } k \in N.
\]

\[\Rightarrow (\Delta_m \bar{x}^j_k) \text{ converges in } R \text{ for all } k \in N.
\]

Let \( \lim_{j} \Delta_m \bar{x}^j_k = \bar{y}_k \), for each \( k \in N \).

Since \( \lim_{j} \bar{x}^j_k = \bar{x}_k \), for each \( k \in N \), therefore \( \lim_{j} \bar{x}^j_k = \bar{x}_k \) exist for each \( k \in N \).

We have
\[
\lim_{j \rightarrow \infty} \sum_{k=1}^{\infty} d(\bar{x}^i_k, \bar{x}^j_k) = \sum_{k=1}^{\infty} d(\bar{x}^i_k, \bar{x}_k) < \varepsilon, \text{for all } i \geq n_0
\]
and
\[
\lim_{j \rightarrow \infty} d \left( (\bar{x}^i_{k+m} - \bar{x}^j_{k+m}), (\bar{x}^i_k - \bar{x}^j_k) \right) = d \left( (\bar{x}^i_{k+m} - \bar{x}_{k+m}), (\bar{x}^i_k - \bar{x}_m) \right) < \varepsilon,
\]
for all \( i \geq n_0 \) and \( k \in N \).

Hence for all \( i \geq n_0 \), \( \sup_{k} d(\Delta_m \bar{x}^i_k, \Delta_m \bar{x}_k) < \varepsilon \).

Thus we have
\[
\sum_{k=1}^{\infty} [d(\bar{x}^i_k, \bar{x}_k)]^{p_k} + \sup_{k} d(\Delta_m \bar{x}^i_k, \Delta_m \bar{x}_k) < 2\varepsilon, \text{for all } i \geq n_0.
\]

\[\Rightarrow \rho(\bar{x}^i, \bar{x}) < 2\varepsilon, \text{for all } i \geq n_0.
\]

i.e. \( \bar{x}^i \rightarrow \bar{x} \), as \( i \rightarrow \infty \) in \( \tilde{l}(p)(\Delta_m) \).

And for \( i \geq n_0 \),
\[
\sup_{k} \left( d(\Delta_m \bar{x}_k, \bar{0}) \right) \leq \sup_{k} \left( d(\Delta_m \bar{x}_k, \Delta_m \bar{x}^i_k) \right) + \sup_{k} \left( d(\Delta_m \bar{x}^i_k, \bar{0}) \right) < \infty.
\]

This completes the proof.
**Theorem 2.2:** The sequence spaces $\ell(p)(\Delta_m), c(p)(\Delta_m)$ and $c_0(p)(\Delta_m)$ are solid.

**Proof:** Let $\bar{x} = (\bar{x}_k) \in \ell(p)(\Delta_m)$ and $\bar{y} = (\bar{y}_k) \in \ell(p)(\Delta_m)$ be interval valued sequences such that $|\bar{y}_k| \leq |\bar{x}_k|$ for all $k \in N$.

Then

$$\sum_{k=1}^{\infty} [d(\Delta_m \bar{x}_k, \bar{0})]^p_k < \infty$$

and

$$\sum_{k=1}^{\infty} [d(\Delta_m \bar{y}_k, \bar{0})]^p_k \leq \sum_{k=1}^{\infty} [d(\Delta_m \bar{x}_k, \bar{0})]^p_k < \infty.$$ 

Thus $\bar{y} = (\bar{y}_k) \in \ell(p)(\Delta_m)$ and hence $\ell(p)(\Delta_m)$ is solid. This completes the proof.

**Theorem 2.3:** The sequence spaces $\ell(p)(\Delta_m), c(p)(\Delta_m)$ and $c_0(p)(\Delta_m)$ are not convergence free.

**Proof:** Let $m = 2$, we consider the interval sequence $\bar{x} = (\bar{x}_k)$ as follows

$$\bar{x}_k = \left[ \frac{-1}{k^2}, 0 \right], \Delta_2 \bar{x} = \left[ \frac{-1}{k^2}, \frac{1}{(k+2)^2} \right], \text{ for all } k \in N.$$ 

Then, for $p_k = 1$

$$\sum_{k=1}^{\infty} [d(\Delta_2 \bar{x}_k, \bar{0})] < \sum_{k=1}^{\infty} \left( \frac{1}{k^2} \right) < \infty.$$ 

Thus $\bar{x} = (\bar{x}_k) \in \ell(p)(\Delta_m)$.

Now let us define $\bar{y} = (\bar{y}_k)$ as follows

$$\bar{y}_k = [-k^2, 0], \text{ then } \Delta_2 \bar{y}_k = [-k^2, (k+2)^2], \text{ for all } k \in N.$$ 

Then

$$\sum_{k=1}^{\infty} [d(\Delta_2 \bar{y}_k, \bar{0})] \leq \sum_{k=1}^{\infty} (k+2)^2 = \infty.$$ 

Thus $\bar{y} = (\bar{y}_k) \notin \ell(p)(\Delta_m)$.

Hence $\ell(p)(\Delta_m)$ is not convergence free.

This completes the proof.
References


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