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Star edge coloring of corona product of path and wheel graph families

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Abstract

A star edge coloring of a graph G is a proper edge coloring without bichromatic paths and cycles of length four. In this paper, we obtain the star edge chromatic number of the corona product of path with cycle, path with wheel, path with helm and path with gear graphs, denoted by $P_m \circ C_n$, $P_m \circ W_n$, $P_m \circ H_n$, $P_m \circ G_n$ respectively.

Keywords : *Star edge coloring, corona graph, path, cycle, wheel, helm and gear graph.*

1. Introduction

All graphs considered in this paper are finite and simple, i.e. undirected, loopless and without multiple edges. The Maximum degree of a graph G is denoted by Δ.

The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where the i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

For any integer $n \geq 4$, the wheel graph W_n is the $n-$vertex graph obtained by joining a vertex v_1 to each of the $n-1$ vertices $\{w_1, w_2, \ldots w_{n-1}\}$ of the cycle graph C_{n-1}.

The helm graph H_n is the graph obtained from an $(n+1)$-wheel graph by adjoining a pendent edge at each node of the n-cycle.

The gear graph G_n, also known as a bipartite wheel graph, is a $(n+1)$-wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle.

An edge coloring of graph $G = (V, E)$ is a function $C : E \to N$, in which any two adjacent edges $e, f \in E$ are assigned different colors. The function C is known as the edge-coloring function. A graph G for which there exists an edge-coloring which requires k colors is called k-edge colorable, while such a coloring is called a k-edge coloring. The smallest number k of which there exists a k-edge-coloring of G is called the chromatic index of a graph G and is denoted by $\chi'(G)$.

A star edge coloring of a graph G is a proper edge coloring where at least three distinct colors are used on the edges of every path and cycle of length four, i.e., there is neither bichromatic path nor cycle of length four. The minimum number of colors for which G admits a star edge coloring is called the star edge chromatic index and it is denoted by $\chi'_{st}(G)$.

The star edge coloring was initiated in 2008 by Liu and Deng [8], motivated by the vertex version (see [1, 3, 4, 6, 7, 10]). Dvořák, Mohar and Šámal [5] determined upper and lower bounds for complete graphs. L'udmila Bezegová et.al [9] discussed the star edge chromatic number of trees and outerplanar graphs in terms of its maximum degree Δ.

Additional graph theory terminology used in this paper can be found in [2].

In the following section, we discuss the star edge chromatic number of path with cycle, path with wheel, path with helm and path with gear graphs, denoted by $P_m \circ C_n$, $P_m \circ W_n$, $P_m \circ H_n$, $P_m \circ G_n$ respectively.

2. Main Results

Theorem 2.1. *For any positive integer m and $n > 4$, then*

$$\chi'_{st}\left(P_m \circ C_n\right) = \Delta.$$

Proof. Let $V\left(P_m\right) = \{u_i : 1 \leq i \leq m\}$ and

$V\left(C_n\right) = \{v_j : 1 \leq j \leq n\}$. Let $E\left(P_m\right) = \{u_i u_{i+1} : 1 \leq i \leq m-1\}$ and $E\left(C_n\right) = \{v_j v_{j+1} : 1 \leq j \leq n-1\} \cup \{v_n v_1\}$. By the definition of corona graph,

$$\begin{aligned} V\left(P_m \circ C_n\right) &= V\left(P_m\right) \cup \bigcup_{i=1}^{m} \{v_{ij} : 1 \leq j \leq n\} \text{ and} \\ E\left(P_m \circ C_n\right) &= E\left(P_m\right) \cup \bigcup_{i=1}^{m} \{u_i v_{ij} : 1 \leq j \leq n\} \cup \bigcup_{i=1}^{m} \{v_{ij} v_{ij+1} : 1 \leq j \leq n-1\} \\ &\quad \cup \bigcup_{i=1}^{m} \{v_{in-1} v_{i1}\}. \end{aligned}$$

Let f be a mapping from $E\left(P_m \circ C_n\right)$ as follows:

Case 1: If $m \geq 3$.
For $1 \leq i \leq m$,

$$f\left(u_i v_{ij}\right) = j, 1 \leq j \leq n-1; \quad (2.1)$$

$$f\left(v_{ij} v_{ij+1}\right) = \begin{cases} j+3 \,(\operatorname{mod} n) & \text{if} \quad j+3 \not\equiv 0 \,(\operatorname{mod} n) \\ n \,(\operatorname{mod} n) & \text{if} \quad j+3 \equiv 0 \,(\operatorname{mod} m+n); \end{cases}$$

(2.2)

$f\left(u_{3i-2} u_{3i-1}\right) = n+1,\ 1 \leq i \leq \left\lceil \frac{m-1}{3} \right\rceil$; $f\left(u_{3i-1} u_{3i}\right) = n+2,\ 1 \leq i \leq \left\lceil \frac{m-1}{3} \right\rceil$; $f\left(u_{3i} u_{3i+1}\right) = n,\ 1 \leq i \leq \left\lceil \frac{m-1}{3} \right\rceil$; $f\left(u_{3i-2} v_{3i-2,n}\right) = n+2,\ 1 \leq i \leq \left\lceil \frac{m-1}{3} \right\rceil$; $f\left(u_{3i-1} v_{3i-1,n}\right) = n,\ 1 \leq i \leq \left\lceil \frac{m-1}{3} \right\rceil$; $f\left(u_{3i} v_{3i,n}\right) = n,\ 1 \leq i \leq \left\lfloor \frac{m}{3} \right\rfloor$.

Case 2: If $m = 2$.

$$f(u_i v_{ij}) = j,\ 1 \le i \le m, 1 \le j \le n; \tag{2.3}$$

$f(u_1 u_2) = n + 1$; and using equation (2.1) and (2.2).

Case 3: If $m = 1$, determine f using equation (2.1), (2.2) and (2.3).

It is easy to see that f satisfies no bichromatic 4-path. We assume that $\chi'_{st}(P_m \circ C_n) \le \Delta$. We know that $\chi'_{st}(P_m \circ C_n) \ge \chi'(P_m \circ C_n) \ge \Delta$, since $\chi'_{st}(P_m \circ C_n) \ge \Delta$. Therefore $\chi'_{st}(P_m \circ C_n) = \Delta$. □

Theorem 2.2. *For any positive integer m and $n > 4$, then*

$$\chi'_{st}(P_m \circ W_n) = \Delta.$$

Proof. Let $V(P_m) = \{u_i : 1 \le i \le m\}$ and $V(W_n) = \{v_n\} \cup \{v_j : 1 \le j \le n-1\}$. Let $E(P_m) = \{u_i u_{i+1} : 1 \le i \le m-1\}$ and $E(W_n) = \{v_n v_j : 1 \le j \le n-1\} \cup \{v_j v_{j+1} : 1 \le j \le n-2\} \cup \{v_{n-1} v_1\}$. By the definition of corona graph,

$$\begin{aligned} V(P_m \circ W_n) &= V(P_m) \cup \bigcup_{i=1}^{m} \{v_{ij} : 1 \le j \le n\} \text{ and} \\ E(P_m \circ W_n) &= E(P_m) \cup \bigcup_{i=1}^{m} \{u_i v_{ij} : 1 \le j \le n\} \cup \bigcup_{i=1}^{m} \{v_{in} v_{ij} : 1 \le j \le n-1\} \\ &\quad \cup \bigcup_{i=1}^{m} \{v_{ij} v_{ij+1} : 1 \le j \le n-1\} \cup \bigcup_{i=1}^{m} \{v_{in-1} v_{i1}\}. \end{aligned}$$

Let f be a mapping from $E(P_m \circ W_n)$ as follows:

Case 1: If $m \ge 3$.

$$\left\{ \begin{array}{l} \text{For } 1 \le i \le m, \\ f(u_i v_{ij}) = j, 1 \le j \le n-1; f(v_{in} v_{ij}) = j+1, 1 \le j \le n-2; \\ f(v_{in} v_{in-1}) = 1; f(v_{ij} v_{ij+1}) = j+3, 1 \le j \le n-2; \end{array} \right.$$

(2.4)

$f(v_{in-1}v_{i1}) = n+2, 1 \le i \le m$; $f(u_{3i-2}u_{3i-1}) = n$, $1 \le i \le \left\lceil \frac{m-1}{3} \right\rceil$; $f(u_{3i-1}u_{3i}) = n+1$, $1 \le i \le \left\lceil \frac{m-1}{3} \right\rceil$; $f(u_{3i}u_{3i+1}) = n+2$, $1 \le i \le \left\lceil \frac{m-1}{3} \right\rceil$; $f(u_{3i-2}v_{3i-2n}) = n+1$, $1 \le i \le \left\lceil \frac{m}{3} \right\rceil$; $f(u_{3i-1}v_{3i-1n}) = n+2$, $1 \le i \le \left\lfloor \frac{m}{3} \right\rfloor$; $f(u_{3i}v_{3in}) = n$, $1 \le i \le \left\lfloor \frac{m}{3} \right\rfloor$.

Case 2: If $m = 2$.
$f(u_1u_2) = n+1$; $f(v_{in-1}v_{i1}) = 3$; and using equation (2.4).

Case 3: If $m = 1$.
$f(v_{1n-1}v_{11}) = 3$; and using equation (2.4).

Clearly the above color partitions satisfies no bichromatic 4-path. We assume that $\chi'_{st}(P_m \circ W_n) \le \Delta$. We know that $\chi'_{st}(P_m \circ W_n) \ge \chi'(P_m \circ W_n) \ge \Delta$, since $\chi'_{st}(P_m \circ W_n) \ge \Delta$. Therefore $\chi'_{st}(P_m \circ W_n) = \Delta$. □

Theorem 2.3. *For any positive integer m and $n > 4$, then*

$$\chi'_{st}(P_m \circ H_n) = \Delta.$$

Proof. Let $V(P_m) = \{u_i : 1 \le i \le m\}$ and $V(H_n) = \{v_n\} \cup \{v_j : 1 \le j \le n-1\} \cup \left\{v'_j : 1 \le j \le n-1\right\}$. Let $E(P_m) = \{u_iu_{i+1} : 1 \le i \le m-1\}$ and $E(H_n) = \{v_nv_j : 1 \le j \le n-1\} \cup \left\{v_jv'_j : 1 \le j \le n-1\right\} \cup \{v_jv_{j+1} : 1 \le j \le n-2\} \cup \{v_{n-1}v_1\}$. By the definition of corona graph,

$$\begin{aligned} V(P_m \circ H_n) &= V(P_m) \cup \bigcup_{i=1}^{m} \{v_{in}\} \cup \bigcup_{i=1}^{m} \{v_{ij} : 1 \le j \le n-1\} \cup \bigcup_{i=1}^{m} \left\{v'_{ij} : 1 \le j \le n-1\right\}, \\ E(P_m \circ H_n) &= E(P_m) \cup \bigcup_{i=1}^{m} \{u_iv_{ij} : 1 \le j \le n\} \cup \bigcup_{i=1}^{m} \left\{u_iv'_{ij} : 1 \le j \le n-1\right\} \\ &\quad \cup \bigcup_{i=1}^{m} \{v_{in}v_{ij} : 1 \le j \le n-1\} \cup \bigcup_{i=1}^{m} \left\{v_{ij}v'_{ij} : 1 \le j \le n-1\right\} \\ &\quad \cup \bigcup_{i=1}^{m} \{v_{ij}v_{i,j+1} : 1 \le j \le n-2\} \cup \bigcup_{i=1}^{m} \{v_{i,n-1}v_{i1}\}. \end{aligned}$$

Let f be a mapping from $E(P_m \circ H_n)$ as follows:

Case 1: If $m \geq 3$.

$$\begin{cases} \text{For } 1 \leq i \leq m, \\ f\left(u_i v_{ij}\right) = j, 1 \leq j \leq n; f\left(u_i v'_j\right) = n + j, 1 \leq j \leq n-2; \\ f\left(v_{in} v_{ij}\right) = j+1, 1 \leq i \leq n-2; f\left(v_{in} v_{in-1}\right) = 1; \\ f\left(v_{ij} v_{ij+1}\right) = n + j - 1, 1 \leq j \leq n-2; f\left(v_{in-1} v_{i1}\right) = 2n-2; \\ f\left(v_{ij} v'_{ij}\right) = n + j + 1, 1 \leq j \leq n-2; \end{cases}$$

(2.5)

$f\left(u_{3i-2} v'_{3i-2,n-1}\right) = 2n+1,\ 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil$; $f\left(u_{3i-1} v'_{3i-1,n-1}\right) = 2n-1,\ 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil$; $f\left(u_{3i} v'_{3i,n-1}\right) = 2n,\ 1 \leq i \leq \left\lfloor \frac{m}{3} \right\rfloor$; $f\left(v_{3i-2,n-1} v'_{3i-2,n-1}\right) = 2n,\ 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil$; $f\left(v_{3i-1,n-1} v'_{3i-1,n-1}\right) = 2n+1,\ 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil$; $f\left(v_{3i,n-1} v'_{3i,n-1}\right) = 2n-1,\ 1 \leq i \leq \left\lfloor \frac{m}{3} \right\rfloor$; $f\left(u_{3i-2} u_{3i-1}\right) = 2n,\ 1 \leq i \leq \left\lceil \frac{m-1}{3} \right\rceil$; $f\left(u_{3i-1} u_{3i}\right) = 2n+1,\ 1 \leq i \leq \left\lfloor \frac{m-1}{3} \right\rfloor$; $f\left(u_{3i} u_{3i+1}\right) = 2n-1,\ 1 \leq i \leq \left\lceil \frac{m-1}{3} \right\rceil$.

Case 2: If $m = 2$.
$f\left(u_1 u_2\right) = 2n$; $f\left(v_{1,n-1} v'_{1,n-1}\right) = 2n$; $f\left(u_1 v'_{1,n-1}\right) = 2n+1$; $f\left(u_2 v'_{2,n-1}\right) = 2n-1$ and using equation (2.5).

Case 3: If $m = 1$.
$f\left(v_{1,n-1} v'_{1,n-1}\right) = 2n$; $f\left(u_1 v'_{1,n-1}\right) = 2n+1$ and using equation (2.5).

Clearly the above color partitions satisfies no bichromatic 4-path. We assume that $\chi'_{st}\left(P_m \circ H_n\right) \leq \Delta$. We know that $\chi'_{st}\left(P_m \circ H_n\right) \geq \chi'\left(P_m \circ H_n\right) \geq \Delta$, since $\chi'_{st}\left(P_m \circ H_n\right) \geq \Delta$. Therefore $\chi'_{st}\left(P_m \circ H_n\right) = \Delta$. □

Theorem 2.4. *For any positive integer m and $n \geq 5$, then*

$$\chi'_{st}\left(P_m \circ G_n\right) = \Delta.$$

Proof. Let $V(P_m) = \{u_i : 1 \le i \le m\}$ and $V(G_n) = \{v_n\} \cup \{v_j : 1 \le j \le n-1\} \cup \left\{v'_j : 1 \le j \le n-1\right\}$. Let $E(P_m) = \{u_i u_{i+1} : 1 \le i \le m-1\}$ and $E(G_n) = \{v_n v_j : 1 \le j \le n-1\} \cup \left\{v_j v'_j : 1 \le j \le n-1\right\} \cup \left\{v'_j v_{j+1} : 1 \le j \le n-1\right\} \cup \left\{v'_{n-1} v_1\right\}$. By the definition of corona graph,

$$
\begin{aligned}
V(P_m \circ G_n) &= V(P_m) \cup \bigcup_{i=1}^{m} \{v_{in}\} \cup \bigcup_{i=1}^{m} \{v_{ij} : 1 \le j \le n-1\} \cup \bigcup_{i=1}^{m} \left\{v'_{ij} : 1 \le j \le n-1\right\}, \\
E(P_m \circ G_n) &= E(P_m) \cup \bigcup_{i=1}^{m} \{u_i v_{ij} : 1 \le j \le n\} \cup \bigcup_{i=1}^{m} \left\{u_i v'_{ij} : 1 \le j \le n-1\right\} \\
&\quad \cup \bigcup_{i=1}^{m} \{v_{in} v_{ij} : 1 \le j \le n-1\} \cup \bigcup_{i=1}^{m} \left\{v_{ij} v'_{ij} : 1 \le j \le n-1\right\} \\
&\quad \cup \bigcup_{i=1}^{m} \left\{v'_{ij} v_{i,j+1} : 1 \le j \le n-2\right\} \cup \bigcup_{i=1}^{m} \left\{v'_{i,n-1} v_{i1}\right\}.
\end{aligned}
$$

Let f be a mapping from $E(P_m \circ G_n)$ as follows:

$$
\begin{cases}
\text{For } 1 \le i \le m, \\
f(u_i v_{ij}) = j, 1 \le j \le n; f\left(u_i v'_{ij}\right) = n + j, 1 \le j \le n-2; \\
f(v_{in} v_{ij}) = j + 1, 1 \le i \le n-2; f(v_{in} v_{in-1}) = 1; \\
f\left(v_{ij} v'_{ij}\right) = j + 2, 1 \le j \le n-1; f\left(v'_{ij} v'_{ij+1}\right) = n + j + 1, 1 \le j \le n-2; \\
f\left(v'_{i,n-1} v_{i1}\right) = 2n;
\end{cases}
\tag{2.6}
$$

$f\left(u_{3i-2} v'_{3i-2,n-1}\right) = 2n+1$, $1 \le i \le \left\lceil \frac{m}{3} \right\rceil$; $f\left(u_{3i-1} v'_{3i-1,n-1}\right) = 2n-1$, $1 \le i \le \left\lceil \frac{m}{3} \right\rceil$; $f\left(u_{3i} v'_{3i,n-1}\right) = 2n$, $1 \le i \le \left\lfloor \frac{m}{3} \right\rfloor$; $f(u_{3i-2} u_{3i-1}) = 2n$, $1 \le i \le \left\lceil \frac{m-1}{3} \right\rceil$; $f(u_{3i-1} u_{3i}) = 2n+1$, $1 \le i \le \left\lfloor \frac{m-1}{3} \right\rfloor$; $f(u_{3i} u_{3i+1}) = 2n-1$, $1 \le i \le \left\lceil \frac{m-1}{3} \right\rceil$.

Clearly the above color partitions satisfies no bichromatic 4-path. We assume that $\chi'_{st}(P_m \circ G_n) \le \Delta$. We know that $\chi'_{st}(P_m \circ G_n) \ge \chi'(P_m \circ G_n) \ge \Delta$, since $\chi'_{st}(P_m \circ G_n) \ge \Delta$. Therefore $\chi'_{st}(P_m \circ G_n) = \Delta$. □

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