

## **Fekete-Szego problem for certain analytic functions defined by $q$ -derivative operator with respect to symmetric and conjugate points**

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### **Abstract**

*Recently, the  $q$ -derivative operator has been used to investigate several subclasses of analytic functions in different ways with different perspectives by many researchers and their interesting results are too voluminous to discuss. For example, the extension of the theory of univalent functions can be used to describe the theory of  $q$ -calculus,  $q$ -calculus operator are also used to construct several subclasses of analytic functions and so on. In this work, we considered the Fekete-Szego problem for certain analytic functions defined by  $q$ -derivative operator with respect to symmetric and conjugate points. The early few coefficient bounds were obtained to derive our results.*

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## 1. Introduction

Let  $A$  represent the class of analytic functions  $f$  defined by the unit disc  $U = \{z : |z| < 1\}$  and given by

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.$$

Also, let  $S \subset A$  consisting of functions which are univalent inside  $U$ . The well-known classes of analytic functions are starlike and convex functions which their geometrical condition satisfies  $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$  and  $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$ .

Let  $S_s^*$  be the subclass of  $S$  consisting of functions given by (1.1) satisfying the condition

$$(1.2) \quad Re\left(\frac{zf'(z)}{f(z) - f(-z)}\right) > 0 \quad z \in U.$$

This class of functions referred to the class of starlike function with respect to symmetric points.

Let  $S_c^*$  be the subclass of  $S$  consisting of functions given by (1.1) satisfying the condition

$$(1.3) \quad Re\left(\frac{zf'(z)}{f(z) - f(\bar{z})}\right) > 0 \quad z \in U.$$

This class of functions referred to the class of starlike function with respect to conjugate points.

Let  $S_s^c$  be the subclass of  $S$  consisting of functions given by (1.1) satisfying the condition

$$(1.4) \quad Re\left(\frac{(zf'(z))'}{(f(z) - f(-z))'}\right) > 0 \quad z \in U.$$

This class of functions referred to the class of convex function with respect to symmetric points.

Let  $S_c^c$  be the subclass of  $S$  consisting of functions given by (1.1) satisfying the condition

$$(1.5) \quad \operatorname{Re} \frac{(zf'(z))'}{(f(z) - f(\bar{z}))'} > 0 \quad z \in U.$$

This class of functions referred to the class of convex function with respect to conjugate points.

Two analytic functions are said to be subordinate to each other written as  $f \prec g$ , if there exists a Schwartz function  $\omega$  which is analytic in  $U$  with  $\omega(0) = 0$  and  $|\omega(z)| < 1$ , for all  $z \in U$ , such that  $f(z) = g(\omega(z))$ , and  $f(U) \subset g(U)$ .

Goel and Mehrotra [4] introduced a subclass of  $S_s^*$  denoted by  $S_s^*(A, B)$  in terms of subordination which their geometrical condition satisfies

$$(1.6) \quad \operatorname{Re} \frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz} \quad -1 \leq B < A \leq 1 \quad z \in U.$$

For a subclass of  $S_c^c$  denoted by  $S_c^c(A, B)$  in terms of subordination and their geometrical condition satisfies

$$(1.7) \quad \operatorname{Re} \frac{2zf'(z)}{f(z) - f(\bar{z})} \prec \frac{1 + Az}{1 + Bz} \quad -1 \leq B < A \leq 1 \quad z \in U.$$

Also, a subclass of  $S_s^c$  denoted by  $S_s^c(A, B)$  in terms of subordination and their geometrical condition satisfies

$$(1.8) \quad \operatorname{Re} \frac{2(zf'(z))'}{(f(z) - f(-z))'} \prec \frac{1 + Az}{1 + Bz} \quad -1 \leq B < A \leq 1 \quad z \in U.$$

Lastly, a subclass of  $S_c^c$  denoted by  $S_c^c(A, B)$  in terms of subordination and their geometrical condition satisfies

$$(1.9) \quad \operatorname{Re} \frac{2(zf'(z))'}{(f(z) - f(\bar{z}))'} \prec \frac{1 + Az}{1 + Bz} \quad -1 \leq B < A \leq 1 \quad z \in U.$$

See details in [3], [4] and [14].

For  $q \in (0, 1)$ , Jackson's  $q$ -derivative [[6],[7]] of a function  $f \in A$  is given by

$$(1.10) \quad (D_q f)(z) = \begin{cases} \frac{f(z) - f(qz)}{(1 - q)z}, & \text{for } z \neq 0 \\ f'(0), & \text{for } z = 0 \end{cases}$$

where  $D_q^2 f(z) = D_q(D_q f(z))$ . It follows from 1a that

$$(1.11) \quad (D_q f)(z) = 1 + \sum_{n=1}^{\infty} [n]_q a_n z^{n-1}, \text{ where } [n]_q = \frac{1 - q^n}{1 - q}$$

and it is called the basic number  $n$ . If  $q \rightarrow 1^-$  then  $[n]_q \rightarrow n$ . See details [2], [11], [12] and [13].

**Definition A :** Let  $P$  denote the class of Caratheodory functions with positive real part in  $U$ . Also, let  $P(p_k)$  where  $k \in [0, \infty)$  denote the family of functions  $P$ , such that  $p \in P$  and  $p \prec p_k$  in  $U$ , and the function  $p_k$  maps the unit disk conformally onto the region  $\Omega_k$  such that  $1 \in \Omega_k$  and

$$(1.12) \quad \partial\Omega_k = \left\{ u + iv : u = k\sqrt{(u-1)^2 + v^2} \right\}.$$

It is noted that, the domain  $\Omega_k$  is elliptic for  $k > 1$ , parabola for  $k = 1$  and hyperbola when  $k \in (0, 1)$ .

Kanas and Wisniowska [9] and [10] shown that the extremal functions  $p_k(z)$  for conic region are given below:

For  $k = 0$ , we have

$$(1.13) \quad p_0(z) = \frac{1+z}{1-z} = 1 + 2z + 2z^2 + 2z^3 + 2z^4 + \dots$$

For  $k = 1$

$$(1.14) \quad p_1(z) = 1 + \frac{2}{\pi} \log^2 \left( \frac{1 + \sqrt{z}}{1 - \sqrt{z}} \right) = 1 + \frac{8}{\pi^2} z + \frac{16}{3\pi^2} z^2 + \frac{184}{45\pi^2} z^3$$

and lastly for  $k \in (0, 1)$  and  $A = A(k) = \frac{2}{\pi} \arccos k$  we obtain  $p_k(z) = 1 + \frac{2}{1-k^2} \sinh^2(A(k) \operatorname{arctanh} \sqrt{z}) = \frac{1}{1-k^2} \cos \left\{ A(k) i \log \frac{1+\sqrt{z}}{1-\sqrt{z}} - \frac{k^2}{1-k^2} \right\}$

$$(1.15) \quad 1 + \frac{2A^2}{1-k^2} z + \frac{4A^2 + 2A^4}{3(1-k^2)} z^2 + \frac{\frac{46A^2}{15} + \frac{8A^4}{3} + \frac{4A^6}{15}}{3(1-k^2)} z^3 + \dots$$

and for  $k > 1$  and  $u(z) = \frac{z-\sqrt{k}}{1-\sqrt{k}z}$  we have

$$p_k(z) = \frac{1}{k^2 - 1} \sin \left( \frac{\pi}{2K(k)} \int_0^{\frac{u(z)}{\sqrt{k}}} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}} \right)$$

$$(1.16) \quad = 1 + \frac{\pi^2}{4\sqrt{k(k^2-1)}K^2(k)(1+k)} \left\{ z + \frac{4K^2(k)(k^2+6k+1) - \pi^2}{4\sqrt{k(k^2-1)}K^2(k)(1+k)} z^2 + \dots \right\}$$

where  $K(k)$  indicates the Legendre's complete elliptic integral of the first kind and  $K'(k)$  is the complementary integrand of  $K(k)$  with  $k \in (0, 1)$  is chosen such that  $k = \cosh[(\pi K'(k)/4K(k))]$ . By virtue of  $p(z) = \frac{zf'(z)}{f(z)} \prec p_k(z)$  or  $p(z) = 1 + \frac{zf''(z)}{f'(z)} \prec p_k(z)$  and the properties of the domains, we have

$$R(p(z)) > R(p_k(z)) > \frac{k}{k+1}.$$

Motivated by earlier work done by [1], [3], [4], [5], [8] and [14]. In this present investigation, the Fekete-Szego problem for certain classes of analytic function defined by  $q$ -derivative operator with respect to symmetric and conjugate points were considered. The first few coefficients bound for the classes were obtained which serve as a tools for our results. Sequel to the next section, the following Lemma and definitions shall be necessary.

**Lemma 1.1.** [8] Let  $\omega(z) = \omega_1 z + \omega^2 z^2 + \dots \in U$  be so that  $|\omega(z)| < 1$  in  $U$ . If  $t$  is a complex number then

$$(1.17) \quad |\omega^2 + t\omega_1^2| \leq \max\{1, |t|\}.$$

The inequality is sharp for the functions  $\omega(z) = z$  or  $\omega(z) = z^2$ .

We shall employ [4] to give the following definition.

**Definition 1.1.** For the real numbers  $k \in [0, \infty)$ ,  $q \in (0, 1)$  and for  $p_k(z)$  as in Definition A, we say that a function  $f \in A$  is in the class  $S_s^*(p_k)$  if

$$(1.18) \quad \frac{2z(D_q f(z))}{f(z) - f(-z)} \prec p_k(z).$$

**Definition 1.2.** For the real numbers  $k \in [0, \infty)$ ,  $q \in (0, 1)$  and for  $p_k(z)$  as in Definition A, we say that a function  $f \in A$  is in the class  $S_c^*(p_k)$  if

$$(1.19) \quad \frac{2z(D_q f(z))}{f(z) + \overline{f(\overline{z})}} \prec p_k(z).$$

**Definition 1.3.** For the real numbers  $k \in [0, \infty)$ ,  $q \in (0, 1)$  and for  $p_k(z)$  as in Definition A, we say that a function  $f \in A$  is in the class  $S_s^c(p_k)$  if

$$(1.20) \quad \frac{2(z(D_q f(z)))'}{(f(z) - f(-z))'} \prec p_k(z).$$

**Definition 1.4.** For the real numbers  $k \in [0, \infty)$ ,  $q \in (0, 1)$  and for  $p_k(z)$  as in Definition A, we say that a function  $f \in A$  is in the class  $S_c^c(p_k)$  if

$$(1.21) \quad \frac{2(z(D_q f(z)))'}{(f(z) + \overline{f(\bar{z})})'} \prec p_k(z).$$

## 2. Main Result

**Theorem 2.1.** Let  $k \in [0, \infty)$ ,  $q \in (0, 1)$  and let  $p_k(z) = 1 + p_1 z + p_2 z^2 + \dots$  be defined as in the Definition A. If  $f$  given by (1.1) belongs to  $S_s^*(p_k)$  then we have

$$(2.1) \quad |a_3 - \mu a_2^2| \leq \frac{p_1}{[3]_q - 1} \max \left\{ 1, \left| \frac{p_2}{p_1} - \frac{\mu p_1 ([3]_q - 1)}{[2]_q^2} \right| \right\}$$

(2.1) holds for any complex number  $\mu$ .

**Proof.** If  $f \in S_s^*(p_k)$ , then there is a Schwarz function  $\omega = \omega_1 z + \omega_2 z^2 + \dots \in U$  such that

$$(2.2) \quad \frac{2z(D_q f(z))}{f(z) - f(-z)} = p_k(\omega(z)).$$

We note that

$$(2.3) \quad 2z(D_q f(z)) = (f(z) - f(-z)) p_k(\omega(z))$$

and

$$p_k(\omega(z)) = 1 + p_1 \omega_1 z + (p_1 \omega_2 + p_2 \omega_1^2) z^2 + (p_1 \omega_3 + 2p_2 \omega_1 \omega_2 + p_3 \omega_1^3) z^3 + \dots$$

$$(2.4)$$

Applying (2.2), (2.3) and (2.4), we obtain

$$(2.5) \quad a_2 = \frac{p_1 \omega_1}{[2]_q}$$

and

$$(2.6) \quad a_3 = \frac{p_1\omega_2}{[3]_q - 1} + \frac{p_2\omega_1^2}{[3]_q - 1} + \frac{p_1^2\omega_1^2}{([2]_q - 1)([3]_q - 1)}.$$

Hence, by (2.5) and (2.6), we get the following

$$(2.7) \quad a_3 - \mu a_2^2 = \frac{p_1}{[3]_q - 1} (\omega_2 + t\omega_1^2)$$

where

$$(2.8) \quad t = \frac{p_2}{p_1} - \frac{\mu p_1([3]_q - 1)}{[2]_q^2}.$$

The result (2.7) now follows from (1.17).  $\square$

Following the proof of Theorem 2.1 and definitions 1.2, 1.3 and 1.4, we have

**Theorem 2.2.** Let  $k \in [0, \infty)$ ,  $q \in (0, 1)$  and let  $p_k(z) = 1 + p_1z + p_2z^2 + \dots$  be defined as in the Definition A. If  $f$  given by (1.1) belongs to  $S_c^*(p_k)$  then we have

$$(2.9) \quad |a_3 - \mu a_2^2| \leq \frac{p_1}{[3]_q - 1} \max \left\{ 1, \left| \frac{p_2}{p_1} + \frac{p_1([2]_q - 1) - \mu p_1([3]_q - 1)}{([2]_q - 1)^2} \right| \right\}$$

(2.9) holds for any complex number  $\mu$ .

**Theorem 2.3.** Let  $k \in [0, \infty)$ ,  $q \in (0, 1)$  and let  $p_k(z) = 1 + p_1z + p_2z^2 + \dots$  be defined as in the Definition A. If  $f$  given by (1.1) belongs to  $S_s^c(p_k)$  then we have

$$(2.10) \quad |a_3 - \mu a_2^2| \leq \frac{p_1}{3[3]_q - 2} \max \left\{ 1, \left| \frac{p_2}{p_1} - \frac{3\mu p_1([3]_q - 2)}{4[2]_q^2} \right| \right\}$$

(2.10) holds for any complex number  $\mu$ .

**Theorem 2.4.** Let  $k \in [0, \infty)$ ,  $q \in (0, 1)$  and let  $p_k(z) = 1 + p_1z + p_2z^2 + \dots$  be defined as in the Definition A. If  $f$  given by (1.1) belongs to  $S_c^c(p_k)$  then we have

$$(2.11) \quad |a_3 - \mu a_2^2| \leq \frac{p_1}{3([3]_q - 1)} \max \left\{ 1, \left| \frac{p_2}{p_1} + \frac{4([2]_q - 1)p_1 - 3\mu p_1([3]_q - 1)}{4([2]_q - 1)^2} \right| \right\}$$

(2.11) holds for any complex number  $\mu$ .

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