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A new type of difference operator Δ^3 on triple sequence spaces

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Abstract

In this paper we have introduced and investigated the difference triple sequence spaces $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$, $\ell_\infty^3(\Delta^3)$ and $c^{3B}(\Delta^3)$ applying the difference operator Δ^3, on the triple sequence (x_{lmn}) and studied some of their algebraic and topological properties. We have also proved some inclusion relation involving these sequence spaces.

Key Words : *Triple sequence space, difference operator, solidity, symmetricity.*

AMS Classification : *40A05; 40B05; 40C05; 46A45.*

1. Introduction

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz [10] for the sequence spaces $c_0(\Delta)$, $c(\Delta)$ and $\ell_\infty(\Delta)$ as follows:

$Z(\Delta) = \{(x_n) \in w : (\Delta x_n) \in Z\}$, for $Z = c_0, c$ and ℓ_∞ the spaces of convergent to zero, convergent and bounded sequences, respectively,

where $(\Delta x) = (\Delta x_n) = (x_n - x_{n+1})$ for all $n \in \mathbf{N}$. The above spaces are Banach Spaces, normed by $\|x\|_\Delta = |x_1| + \sup_n \|\Delta x_n\|$. Et. and Colok [19] generalized this notion as follows:

$(\Delta^p x) = (\Delta^p x_n) = (\Delta^{p-1} x_n - \Delta^{p-1} x_{n+1})$, $\Delta^0 x = x$ and this generalized difference notion has the following binomial representation:

$\Delta^p x_n = \sum_{i=0}^{p} (-1)^i \begin{pmatrix} p \\ i \end{pmatrix} x_{n+i}$ for all $n \in \mathbf{N}$.

The idea of Kizmaz [10] was extended by Et. and Esi. [18], Tripathy [8] and many others. Esi and Tripathy [2] introduded the notion of difference sequence space as $\Delta_m x = (\Delta_m x_n) = x_n - x_{n+m}$ for all $n \in \mathbf{N}$ and $m \in \mathbf{N}$ is fixed. Later on it was studied by Tripathy and Sarma [9], who introduced difference double sequence spaces as follows:

$Z(\Delta) = \{(x_{mn}) \in w : (\Delta x_{mn}) \in Z\}$, for $Z = c^2, c_0^2, \ell_\infty^2$, the spaces of convergent, null and bounded double sequences respectively, where $\Delta x_{mn} = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbf{N}$.

A triple sequence (real or complex) can be defined as a function $x : \mathbf{N} \times \mathbf{N} \times \mathbf{N} \longrightarrow \mathbf{R}(\mathbf{C})$, where $\mathbf{N}$, $\mathbf{R}$ and $\mathbf{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequences was introduced and investigated at the initial stage by Sahiner et. al. [4], Datta et. al. [3], Tripathy and Goswami [13] and many other researchers, see for instant [10, 11, 12]. Savas and Esi [15] have introduced statistical convergence of triple sequences in probabilistic normed space. Later on, Esi [1] have introduced statistical convergence of triple sequences in topological groups. Das [6, 7] have studied I-convergent triple sequence spaces using the modulus function and Six Dimensional Matrix transformation applying the triple Sequences respectively.

Recently Debnath and Das [20] investigated some algebric and topological properties on $c_0^3(\Delta^2)$, $c^3(\Delta^2)$, $c^{3R}(\Delta^2)$, $\ell_\infty^3(\Delta^2)$ and $c^{3B}(\Delta^2)$ the 2^{nd} order difference triple sequence spaces which are convergent to zero in Pringsheim′s sense, convergent in Pringsheim′s sense, regularly convergent, bounded in Pringsheim′s sense, bounded and convergent respectively. They introduced the difference operator Δ^2 on triple sequence (x_{lmn}), defined by

$$\Delta^2 x_{lmn} = x_{lmn} - 2x_{l+1mn} + x_{l+2mn} - 2x_{lm+1n} + 4x_{l+1m+1n} - 2x_{l+2m+1n} + x_{lm+2n} - 2x_{l+1m+2n} + x_{l+2m+2n} - 2x_{lmn+1} + 4x_{l+1mn+1} - 2x_{l+2mn+1} + 4x_{lm+1n+1} - 8x_{l+1m+1n+1} + 4x_{l+2m+1n+1} - 2x_{lm+2n+1} + 4x_{l+1m+2n+1} - 2x_{l+2m+2n+1} + x_{lmn+2} - 2x_{l+1mn+2} + x_{l+2mn+2} - 2x_{lm+1n+2} + 4x_{l+1m+1n+2} - 2x_{l+2m+1n+2} + x_{lm+2n+2} - 2x_{l+1m+2n+2} + x_{l+2m+2n+2}$$

When Δ^2 is replaced by Δ, one the will get the difference triple sequences spaces studied by Debnath,Sarma and Das [21].

2. Definitions and Preliminaries

By convergence of a triple sequence, we mean the convergence in the Pringsheim sense that is a triple sequence (x_{lmn}) is said to be convergent to L in Pringsheim′s sense if for every $\varepsilon > 0$, there exists $\mathbf{N}(\varepsilon) \in \mathbf{N}$ such that.

$|x_{lmn} - L| < \varepsilon$ whenever $l \geq \mathbf{N}$, $m \geq \mathbf{N}$, $n \geq \mathbf{N}$ and we write $lim_{l,m,n \longrightarrow \infty} x_{lmn} = L$.

Note 1 A triple sequence is convergent in Pringsheim′s sense may not be bounded [4].

Definition 2.1. [4] A triple sequence (x_{lmn}) is said to be Cauchy sequence if for every $\varepsilon > 0$, there exists $\mathbf{N}(\varepsilon) \in \mathbf{N}$ such that.

$$|x_{lmn} - x_{pqr}| < \varepsilon \text{ whenever } l \geq p \geq \mathbf{N},\ m \geq q \geq \mathbf{N},\ n \geq r \geq \mathbf{N}.$$

Definition 2.2. [4] A triple sequence (x_{lmn}) is said to be bounded if there exists $M > 0$, such that $|x_{lmn}| < M$ for all $l, m, n \in \mathbf{N}$.

Definition 2.3. [3] A triple sequence (x_{lmn}) is said to be converge regularly if it is convergent in Pringsheim′s sense and in addition the following limits holds:

$$lim_{n\longrightarrow\infty}x_{lmn} = L_{lm}\ (l, m \in \mathbf{N}),$$

$$lim_{m\longrightarrow\infty}x_{lmn} = L_{ln}\ (l, n \in \mathbf{N}),$$

$$lim_{l\longrightarrow\infty}x_{lmn} = L_{mn}\ (m, n \in \mathbf{N}).$$

Let w^3 denote the set of all triple sequence of real numbers. Then the class of triple sequences c_0^3, c^3, ℓ_∞^3, c^{3R} and c^{3B} denotes the triple sequence spaces which are convergent to zero in Pringsheim′s sense, convergent in Pringsheim′s sense, bounded in Pringsheim′s sense, regularly convergent, bounded and convergent respectively.

These classes are all linear spaces.

It is well known that $c_0^3 \subset c^3$, $c^{3R} \subset c^{3B} \subset \ell_\infty^3$ and the inclusion are strict.

Theorem 2.1. The spaces c_0^3, c^3, ℓ_∞^3, c^{3R} and c^{3B} are complete normed linear spaces with the normed.

$$\|x\| = sup_{l,m,n}|x_{lmn}| < \infty$$

Proof. Simple.

Example 2.1 [3] Defined the sequence (x_{lmn}) by

$$x_{lmn} = \begin{cases} mn, & l = 3 \\ nl, & m = 5 \\ lm, & n = 7 \\ 8, & otherwise. \end{cases}$$

Then $(x_{lmn}) \rightarrow 8$ in Pringsheim′s sense but not bounded as well as not regularly convergent.

Example 2.2. Let $x_{lmn} = 1$, for all $l, m, n \in \mathbf{N}$. Then (x_{lmn}) is convergent in Pringsheim′s sense, bounded and regularly convergent.

Definition 2.4. [3] A triple sequence space E is said to be solid if $(\alpha_{lmn}x_{lmn}) \in E$ whenever $(x_{lmn}) \in E$ and for all sequences (α_{lmn}) of scalars with $|\alpha_{lmn}| \leq 1$, for all $l, m, n \in \mathbf{N}$.

Definition 2.5. [3] A triple sequence space E is said to be convergence free if $(y_{lmn}) \in E$, whenever $(x_{lmn}) \in E$ and $x_{lmn} = 0$ implies $y_{lmn} = 0$.

Definition 2.6. [3] A triple sequence space E is said to be symmetric if $(x_{lmn}) \in E$ implies $(x_{\pi(l)\pi(m)\pi(n)}) \in E$, where π is a permutation of $\mathbf{N} \times \mathbf{N} \times \mathbf{N}$.

Definition 2.7. [3] A sequence space E is said to be sequence algebra if x_n, $y_n \in E$ implies $x_n \star y_n \in E$.

Now we introduce the 3^{rd} order difference triple sequence spaces as follows:

$c_0^3(\Delta^3) = \{(x_{lmn}) \in w^3 : (\Delta^3 x_{lmn})$ is regularly null $\}$,

$c^3(\Delta^3) = \{(x_{lmn}) \in w^3 : (\Delta^3 x_{lmn})$ is convergent in Pringsheim's sense $\}$,

$c^{3R}(\Delta^3) = \{(x_{lmn}) \in w^3 : (\Delta^3 x_{lmn})$ is regularly convergent $\}$,

$\ell_\infty^3(\Delta^3) = \{(x_{lmn}) \in w^3 : (\Delta^3 x_{lmn})$ is bounded $\}$,

$c^{3B}(\Delta^3) = \{(x_{lmn}) \in w^3 : (\Delta^3 x_{lmn})$ is convergent in Pringsheim's sense and bounded $\}$.

Here

$\Delta^3 x_{lmn} = x_{lmn} - 3x_{lmn+1} + 3x_{lmn+2} - x_{lmn+3} - 3x_{lm+1n} + 9x_{lm+1n+1} - 9x_{lm+1n+2} + 3x_{lm+1n+3} + 3x_{lm+2n} - 9x_{lm+2n+1} + 9x_{lm+2n+2} - 3x_{lm+2n+3} - x_{lm+3n} + 3x_{lm+3n+1} - 3x_{lm+3n+2} + x_{lm+3n+3} - 3x_{l+1mn} + 9x_{l+1mn+1} - 9x_{l+1mn+2} + 3x_{l+1mn+3} + 9x_{l+1m+1n} - 27x_{l+1m+1n+1} + 27x_{l+1m+1n+2} - 9x_{l+1m+1n+3} - 9x_{l+1m+2n} + 27x_{l+1m+2n+1} - 27x_{l+1m+2n+2} + 9x_{l+1m+2n+3} + 3x_{l+1m+3n} - 9x_{l+1m+3n+1} + 9x_{l+1m+3n+2} - 3x_{l+1m+3n+3} + 3x_{l+2mn} - 9x_{l+2mn+1} + 9x_{l+2mn+2} - 3x_{l+2mn+3} - 9x_{l+2m+1n} + 27x_{l+2m+1n+1} - 27x_{l+2m+1n+2} + 9x_{l+2m+1n+3} + 9x_{l+2m+2n} - 27x_{l+2m+2n+1} + 27x_{l+2m+2n+2} - 9x_{l+2m+2n+3} -$

$3x_{l+2m+3n}+9x_{l+2m+3n+1}-9x_{l+2m+3n+2}+3x_{l+2m+3n+3}-x_{l+3mn}+3x_{l+3mn+1}-3x_{l+3mn+2}+x_{l+3mn+3}+3x_{l+3m+1n}-9x_{l+3m+1n+1}+9x_{l+3m+1n+2}-3x_{l+3m+1n+3}-3x_{l+3m+2n}+9x_{l+3m+2n+1}-9x_{l+3m+2n+2}+3x_{l+3m+2n+3}+x_{l+3m+3n}-3x_{l+3m+3n+1}+3x_{l+3m+3n+2}-x_{l+3m+3n+3}$.

In this paper our aim is to introduce the difference operator Δ^3, on the triple sequence (x_{lmn}).

3. Main Results

We state the following result without proof, since it can be established using standard technique.

Theorem 3.1. The classes of sequences $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$, $\ell_\infty^3(\Delta^3)$ and $c^{3B}(\Delta^3)$ are linear spaces.

Theorem 3.2. The classes of sequences $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$, $\ell_\infty^3(\Delta^3)$ and $c^{3B}(\Delta^3)$ are complete normed linear spaces with the norm

$\|x\| = \sum_{r=1}^3 \sup_{l,m} |x_{lmr}| + \sum_{r=1}^3 \sup_{l,n} |x_{lrn}| + \sum_{r=1}^3 \sup_{m,n} |x_{rmn}| + \sup_{l,m,n} |\Delta^3 x_{lmn}| < \infty$.

Proof. Let (x^i) be a Cauchy sequence in $\ell_\infty^3(\Delta^3)$, where $x^i = (x_{lmn}^i) \in \ell_\infty^3(\Delta^3)$ for each $i \in \mathbf{N}$.

Then we have the following expression,

$\|x^i - x^j\| = \sum_{r=1}^3 \sup_{l,m} |x_{lmr}^i - x_{lmr}^j| + \sum_{r=1}^3 \sup_{l,n} |x_{lrn}^i - x_{lrn}^j| + \sum_{r=1}^3 \sup_{m,n} |x_{rmn}^i - x_{rmn}^j| + \sup_{l,m,n} |\Delta^3 x_{lmn}^i - \Delta^3 x_{lmn}^j| \to 0$ as $i, j \to \infty$.

Therefore, we get $|x_{lmn}^i - x_{lmn}^j| \to 0$, for all $i, j \to \infty$ and each $l, m, n \in \mathbf{N} \times \mathbf{N} \times \mathbf{N}$.

Hence $(x_{lmn}^i) = (x_{lmn}^1, x_{lmn}^2, x_{lmn}^3, ...)$ is a Cauchy sequence in $\mathbf{R}$.

Thus, by the completeness of $\mathbf{R}$, it converges to x_{lmn} say, i.e., there exists

$$\lim_{i\to\infty} x^i_{lmn} = x_{lmn} \text{ for each } l,m,n \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}.$$

Further for each $\varepsilon > 0$, there exists $\mathbf{N} = \mathbf{N}(\varepsilon)$, such that for all $i, j \geq \mathbf{N}$, and for all $l,m,n \in \mathbf{N}\times\mathbf{N}\times\mathbf{N}$

$\sum_{r=1}^{3} |x^i_{lmr} - x^j_{lmr}| < \varepsilon$, $\sum_{r=1}^{3} |x^i_{lrn} - x^j_{lrn}| < \varepsilon$, $\sum_{r=1}^{3} |x^i_{rmn} - x^j_{rmn}| < \varepsilon$
$| \Delta^3 x^i_{lmn} - \Delta^3 x^j_{lmn} |= | (x^i_{lmn} - x^j_{lmn}) - 3(x^i_{lmn+1} - x^j_{lmn+1}) + 3(x^i_{lmn+2} - x^j_{lmn+2}) - (x^i_{lmn+3} - x^j_{lmn+3}) - 3(x^i_{lm+1n} - x^j_{lm+1n}) + 9(x^i_{lm+1n+1} - x^j_{lm+1n+1}) - 9(x^i_{lm+1n+2} - x^j_{lm+1n+2}) + 3(x^i_{lm+1n+3} - x^j_{lm+1n+3}) + 3(x^i_{lm+2n} - x^j_{lm+2n}) - 9(x^i_{lm+2n+1} - x^j_{lm+2n+1}) + 9(x^i_{lm+2n+2} - x^j_{lm+2n+2}) - 3(x^i_{lm+2n+3} - x^j_{lm+2n+3}) - (x^i_{lm+3n} - x^j_{lm+3n}) + 3(x^i_{lm+3n+1} - x^j_{lm+3n+1}) - 3(x^i_{lm+3n+2} - x^j_{lm+3n+2}) + (x^i_{lm+3n+3} - x^j_{lm+3n+3}) - 3(x^i_{l+1mn} - x^j_{l+1mn}) + 9(x^i_{l+1mn+1} - x^j_{l+1mn+1}) - 9(x^i_{l+1mn+2} - x^j_{l+1mn+2}) + 3(x^i_{l+1mn+3} - x^j_{l+1mn+3}) + 9(x^i_{l+1m+1n} - x^j_{l+1m+1n}) - 27(x^i_{l+1m+1n+1} - x^j_{l+1m+1n+1}) + 27(x^i_{l+1m+1n+2} - x^j_{l+1m+1n+2}) - 9(x^i_{l+1m+1n+3} - x^j_{l+1m+1n+3}) - 9(x^i_{l+1m+2n} - x^j_{l+1m+2n}) + 27(x^i_{l+1m+2n+1} - x^j_{l+1m+2n+1}) - 27(x^i_{l+1m+2n+2} - x^j_{l+1m+2n+2}) + 9(x^i_{l+1m+2n+3} - x^j_{l+1m+2n+3}) + 3(x^i_{l+1m+3n} - x^j_{l+1m+3n}) - 9(x^i_{l+1m+3n+1} - x^j_{l+1m+3n+1}) + 9(x^i_{l+1m+3n+2} - x^j_{l+1m+3n+2}) - 3(x^i_{l+1m+3n+3} - x^j_{l+1m+3n+3}) + 3(x^i_{l+2mn} - x^j_{l+2mn}) - 9(x^i_{l+2mn+1} - x^j_{l+2mn+1}) + 9(x^i_{l+2mn+2} - x^j_{l+2mn+2}) - 3(x^i_{l+2mn+3} - x^j_{l+2mn+3}) - 9(x^i_{l+2m+1n} - x^j_{l+2m+1n}) + 27(x^i_{l+2m+1n+1} - x^j_{l+2m+1n+1}) - 27(x^i_{l+2m+1n+2} - x^j_{l+2m+1n+2}) + 9(x^i_{l+2m+1n+3} - x^j_{l+2m+1n+3}) + 9(x^i_{l+2m+2n} - x^j_{l+2m+2n}) - 27(x^i_{l+2m+2n+1} - x^j_{l+2m+2n+1}) + 27(x^i_{l+2m+2n+2} - x^j_{l+2m+2n+2}) - 9(x^i_{l+2m+2n+3} - x^j_{l+2m+2n+3}) - 3(x^i_{l+2m+3n} - x^j_{l+2m+3n}) + 9(x^i_{l+2m+3n+1} - x^j_{l+2m+3n+1}) - 9(x^i_{l+2m+3n+2} - x^j_{l+2m+3n+2}) + 3(x^i_{l+2m+3n+3} - x^j_{l+2m+3n+3}) - (x^i_{l+3mn} - x^j_{l+3mn}) + 3(x^i_{l+3mn+1} - x^j_{l+3mn+1}) - 3(x^i_{l+3mn+2} - x^j_{l+3mn+2}) + (x^i_{l+3mn+3} - x^j_{l+3mn+3}) + 3(x^i_{l+3m+1n} - x^j_{l+3m+1n}) - 9(x^i_{l+3m+1n+1} - x^j_{l+3m+1n+1}) + 9(x^i_{l+3m+1n+2} - x^j_{l+3m+1n+2}) - 3(x^i_{l+3m+1n+3} - x^j_{l+3m+1n+3}) - 3(x^i_{l+3m+2n} - x^j_{l+3m+2n}) + 9(x^i_{l+3m+2n+1} - x^j_{l+3m+2n+1}) - 9(x^i_{l+3m+2n+2} - x^j_{l+3m+2n+2}) + 3(x^i_{l+3m+2n+3} - x^j_{l+3m+2n+3}) + (x^i_{l+3m+3n} - x^j_{l+3m+3n}) - 3(x^i_{l+3m+3n+1} - x^j_{l+3m+3n+1}) + 3(x^i_{l+3m+3n+2} - x^j_{l+3m+3n+2}) - (x^i_{l+3m+3n+3} - x^j_{l+3m+3n+3}) |< \varepsilon,$

and

$\lim_{i\to\infty} \sum_{r=1}^{3} |x^i_{lmr} - x^j_{lmr}| = \sum_r |x^i_{lmr} - x_{lmr}| \le \varepsilon,$

$\lim_{i\to\infty} \sum_{r=1}^{3} |x^i_{lrn} - x^j_{lrn}| = \sum_r |x^i_{lrn} - x_{lrn}| \le \varepsilon,$

$\lim_{i\to\infty} \sum_{r=1}^{3} |x^i_{rmn} - x^j_{rmn}| = \sum_r |x^i_{rmn} - x_{rmn}| \le \varepsilon.$

Now we can write
$\lim_j \mid \Delta^3 x^i_{lmn} - \Delta^3 x^j_{lmn} \mid = \mid (x^i_{lmn} - x_{lmn}) - 3(x^i_{lmn+1} - x_{lmn+1}) + 3(x^i_{lmn+2} - x_{lmn+2}) - (x^i_{lmn+3} - x_{lmn+3}) - 3(x^i_{lm+1n} - x_{lm+1n}) + 9(x^i_{lm+1n+1} - x_{lm+1n+1}) - 9(x^i_{lm+1n+2} - x_{lm+1n+2}) + 3(x^i_{lm+1n+3} - x_{lm+1n+3}) + 3(x^i_{lm+2n} - x_{lm+2n}) - 9(x^i_{lm+2n+1} - x_{lm+2n+1}) + 9(x^i_{lm+2n+2} - x_{lm+2n+2}) - 3(x^i_{lm+2n+3} - x_{lm+2n+3}) - (x^i_{lm+3n} - x_{lm+3n}) + 3(x^i_{lm+3n+1} - x_{lm+3n+1}) - 3(x^i_{lm+3n+2} - x_{lm+3n+2}) + (x^i_{lm+3n+3} - x_{lm+3n+3}) - 3(x^i_{l+1mn} - x_{l+1mn}) + 9(x^i_{l+1mn+1} - x_{l+1mn+1}) - 9(x^i_{l+1mn+2} - x_{l+1mn+2}) + 3(x^i_{l+1mn+3} - x_{l+1mn+3}) + 9(x^i_{l+1m+1n} - x_{l+1m+1n}) - 27(x^i_{l+1m+1n+1} - x_{l+1m+1n+1}) + 27(x^i_{l+1m+1n+2} - x_{l+1m+1n+2}) - 9(x^i_{l+1m+1n+3} - x_{l+1m+1n+3}) - 9(x^i_{l+1m+2n} - x_{l+1m+2n}) + 27(x^i_{l+1m+2n+1} - x_{l+1m+2n+1}) - 27(x^i_{l+1m+2n+2} - x_{l+1m+2n+2}) + 9(x^i_{l+1m+2n+3} - x_{l+1m+2n+3}) + 3(x^i_{l+1m+3n} - x_{l+1m+3n}) - 9(x^i_{l+1m+3n+1} - x_{l+1m+3n+1}) + 9(x^i_{l+1m+3n+2} - x_{l+1m+3n+2}) - 3(x^i_{l+1m+3n+3} - x_{l+1m+3n+3}) + 3(x^i_{l+2mn} - x_{l+2mn}) - 9(x^i_{l+2mn+1} - x_{l+2mn+1}) + 9(x^i_{l+2mn+2} - x_{l+2mn+2}) - 3(x^i_{l+2mn+3} - x_{l+2mn+3}) - 9(x^i_{l+2m+1n} - x_{l+2m+1n}) + 27(x^i_{l+2m+1n+1} - x_{l+2m+1n+1}) - 27(x^i_{l+2m+1n+2} - x_{l+2m+1n+2}) + 9(x^i_{l+2m+1n+3} - x_{l+2m+1n+3}) + 9(x^i_{l+2m+2n} - x_{l+2m+2n}) - 27(x^i_{l+2m+2n+1} - x_{l+2m+2n+1}) + 27(x^i_{l+2m+2n+2} - x_{l+2m+2n+2}) - 9(x^i_{l+2m+2n+3} - x_{l+2m+2n+3}) - 3(x^i_{l+2m+3n} - x_{l+2m+3n}) + 9(x^i_{l+2m+3n+1} - x_{l+2m+3n+1}) - 9(x^i_{l+2m+3n+2} - x_{l+2m+3n+2}) + 3(x^i_{l+2m+3n+3} - x_{l+2m+3n+3}) - (x^i_{l+3mn} - x_{l+3mn}) + 3(x^i_{l+3mn+1} - x_{l+3mn+1}) - 3(x^i_{l+3mn+2} - x_{l+3mn+2}) + (x^i_{l+3mn+3} - x_{l+3mn+3}) + 3(x^i_{l+3m+1n} - x_{l+3m+1n}) - 9(x^i_{l+3m+1n+1} - x_{l+3m+1n+1}) + 9(x^i_{l+3m+1n+2} - x_{l+3m+1n+2}) - 3(x^i_{l+3m+1n+3} - x_{l+3m+1n+3}) - 3(x^i_{l+3m+2n} - x_{l+3m+2n}) + 9(x^i_{l+3m+2n+1} - x_{l+3m+2n+1}) - 9(x^i_{l+3m+2n+2} - x_{l+3m+2n+2}) + 3(x^i_{l+3m+2n+3} - x_{l+3m+2n+3}) + (x^i_{l+3m+3n} - x_{l+3m+3n}) - 3(x^i_{l+3m+3n+1} - x_{l+3m+3n+1}) + 3(x^i_{l+3m+3n+2} - x_{l+3m+3n+2}) - (x^i_{l+3m+3n+3} - x_{l+3m+3n+3}) \mid \le \varepsilon$, for all $i \ge \mathbf{N}$.

Since ε is not dependent on l, m, n
$sup_{l,m,n} \mid (x^i_{lmn} - x_{lmn}) - 3(x^i_{lmn+1} - x_{lmn+1}) + 3(x^i_{lmn+2} - x_{lmn+2}) - (x^i_{lmn+3} - x_{lmn+3}) - 3(x^i_{lm+1n} - x_{lm+1n}) + 9(x^i_{lm+1n+1} - x_{lm+1n+1}) - 9(x^i_{lm+1n+2} - x_{lm+1n+2}) + 3(x^i_{lm+1n+3} - x_{lm+1n+3}) + 3(x^i_{lm+2n} - x_{lm+2n}) - 9(x^i_{lm+2n+1} - x_{lm+2n+1}) + 9(x^i_{lm+2n+2} - x_{lm+2n+2}) - 3(x^i_{lm+2n+3} - x_{lm+2n+3}) - (x^i_{lm+3n} - x_{lm+3n}) + 3(x^i_{lm+3n+1} - x_{lm+3n+1}) - 3(x^i_{lm+3n+2} - x_{lm+3n+2}) + (x^i_{lm+3n+3} -$

$x_{lm+3n+3}) - 3(x^i_{l+1mn} - x_{l+1mn}) + 9(x^i_{l+1mn+1} - x_{l+1mn+1}) - 9(x^i_{l+1mn+2} - x_{l+1mn+2}) + 3(x^i_{l+1mn+3} - x_{l+1mn+3}) + 9(x^i_{l+1m+1n} - x_{l+1m+1n}) - 27(x^i_{l+1m+1n+1} - x_{l+1m+1n+1}) + 27(x^i_{l+1m+1n+2} - x_{l+1m+1n+2}) - 9(x^i_{l+1m+1n+3} - x_{l+1m+1n+3}) - 9(x^i_{l+1m+2n} - x_{l+1m+2n}) + 27(x^i_{l+1m+2n+1} - x_{l+1m+2n+1}) - 27(x^i_{l+1m+2n+2} - x_{l+1m+2n+2}) + 9(x^i_{l+1m+2n+3} - x_{l+1m+2n+3}) + 3(x^i_{l+1m+3n} - x_{l+1m+3n}) - 9(x^i_{l+1m+3n+1} - x_{l+1m+3n+1}) + 9(x^i_{l+1m+3n+2} - x_{l+1m+3n+2}) - 3(x^i_{l+1m+3n+3} - x_{l+1m+3n+3}) + 3(x^i_{l+2mn} - x_{l+2mn}) - 9(x^i_{l+2mn+1} - x_{l+2mn+1}) + 9(x^i_{l+2mn+2} - x_{l+2mn+2}) - 3(x^i_{l+2mn+3} - x_{l+2mn+3}) - 9(x^i_{l+2m+1n} - x_{l+2m+1n}) + 27(x^i_{l+2m+1n+1} - x_{l+2m+1n+1}) - 27(x^i_{l+2m+1n+2} - x_{l+2m+1n+2}) + 9(x^i_{l+2m+1n+3} - x_{l+2m+1n+3}) + 9(x^i_{l+2m+2n} - x_{l+2m+2n}) - 27(x^i_{l+2m+2n+1} - x_{l+2m+2n+1}) + 27(x^i_{l+2m+2n+2} - x_{l+2m+2n+2}) - 9(x^i_{l+2m+2n+3} - x_{l+2m+2n+3}) - 3(x^i_{l+2m+3n} - x_{l+2m+3n}) + 9(x^i_{l+2m+3n+1} - x_{l+2m+3n+1}) - 9(x^i_{l+2m+3n+2} - x_{l+2m+3n+2}) + 3(x^i_{l+2m+3n+3} - x_{l+2m+3n+3}) - (x^i_{l+3mn} - x_{l+3mn}) + 3(x^i_{l+3mn+1} - x_{l+3mn+1}) - 3(x^i_{l+3mn+2} - x_{l+3mn+2}) + (x^i_{l+3mn+3} - x_{l+3mn+3}) + 3(x^i_{l+3m+1n} - x_{l+3m+1n}) - 9(x^i_{l+3m+1n+1} - x_{l+3m+1n+1}) + 9(x^i_{l+3m+1n+2} - x_{l+3m+1n+2}) - 3(x^i_{l+3m+1n+3} - x_{l+3m+1n+3}) - 3(x^i_{l+3m+2n} - x_{l+3m+2n}) + 9(x^i_{l+3m+2n+1} - x_{l+3m+2n+1}) - 9(x^i_{l+3m+2n+2} - x_{l+3m+2n+2}) + 3(x^i_{l+3m+2n+3} - x_{l+3m+2n+3}) + (x^i_{l+3m+3n} - x_{l+3m+3n}) - 3(x^i_{l+3m+3n+1} - x_{l+3m+3n+1}) + 3(x^i_{l+3m+3n+2} - x_{l+3m+3n+2}) - (x^i_{l+3m+3n+3} - x_{l+3m+3n+3}) \mid \leq \varepsilon.$

Consequently we have, $\|x^i_{lmn} - x_{lmn}\| \leq 4\varepsilon$, for all $i \geq \mathbf{N}$.

Therefore we obtain $x^i_{lmn} \to x_{lmn}$ as $i \to \infty$ in $\ell^3_\infty(\Delta^3)$.

Now we show that $(x_{lmn}) \in \ell^3_\infty(\Delta^3)$
$|x_{lmn} - x_{l+3m+3n+3}| = |x_{lmn} - x^N_{lmn} + x^N_{lmn} - x^N_{l+3m+3n+3} + x^N_{l+3m+3n+3} - x_{l+3m+3n+3}|$

$\leq |x^N_{lmn} - x_{l+3m+3n+3}| + \|x^N_{lmn} - x_{lmn}\| = O(1).$

This implies $x = (x_{lmn}) \in \ell^3_\infty(\Delta^3)$.

Since $\ell^3_\infty(\Delta^3)$ is a linear space.

Hence $\ell^3_\infty(\Delta^3)$ is complete.

Similarly the other cases can be established.

Result 3.1.

(i) $c_0^3(\Delta^3) \subset c^3(\Delta^3)$ and the inclusion is strict.

(ii) $c^{3R}(\Delta^3) \subset c^3(\Delta^3)$ and the inclusion is strict.

(iii) $c^{3R}(\Delta^3) \subset c^{3B}(\Delta^3)$ and the inclusion is strict.

Proof. The inclusions can be established easily. The inclusion strict follows from the following examples:

Example 3.1. To prove theorem (i) we consider the sequence (x_{lmn}) defined by

$(x_{lmn})= \frac{1}{3}(-5)^{l+m+n-1}$, for all $l, m, n \in \mathbf{N}$.

Then $(\Delta^3 x_{lmn}) \in c^3$, but the sequence $(\Delta^3 x_{lmn}) \notin c_0^3$.

Hence the inclusion is strict.

Example 3.2. For the case (ii), we consider the sequence defined by

$$(x_{lmn}) = \begin{cases} (-1)^l lmn, & for \quad l \in \mathbf{N}, \quad m = 1, \quad and \quad n = 1, 2, 3 \\ 3, & otherwise. \end{cases}$$

Clearly $(\Delta^3 x_{lmn}) \in c^3$, but the sequence $(\Delta^3 x_{lmn}) \notin c^{3R}$.

Hence the inclusion $c^{3R}(\Delta^3) \subset c^3(\Delta^3)$ is strict.

Example 3.3. For the case (iii), we consider the sequence defined by

$$(x_{lmn}) = \begin{cases} 0, & for \quad l \quad is \quad odd \quad and \quad for\ all\ m, n \in \mathbf{N} \\ lm, & otherwise. \end{cases}$$

Clearly $(\Delta^3 x_{lmn}) \in c^{3B}$, but the sequence $(\Delta^3 x_{lmn}) \notin c^{3R}$.

Hence the inclusion is strict.

Result 3.2. The classes of sequences $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$, $\ell_\infty^3(\Delta^3)$ and $c^{3B}(\Delta^3)$ are not symmetric in general.

Proof. The result follows from the following examples:

Example 3.4. Consider the triple sequence (x_{lmn}) defined by

$x_{lmn} = l$, for all $l, m, n \in \mathbf{N}$.

Clearly the sequence $(\Delta^3 x_{lmn}) \in c_0^3, c^3, c^{3R}$ and c^{3B}.

Consider a rearrange sequence (y_{lmn}) of (x_{lmn}) defined by

$$y_{lmn} = \begin{cases} l+1, & for \quad l = m, \quad n \quad is \quad even \\ l-1, & for \quad l = m+1, \quad n \quad is \quad even \\ l, & otherwise. \end{cases}$$

Clearly $(\Delta^3 y_{lmn}) \notin c_0^3, c^3, c^{3R}$ and c^{3B}.

Hence $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$ and $c^{3B}(\Delta^3)$ are not symmetric.

Example 3.5. We consider the triple sequence (x_{lmn}) defined by

$x_{lmn} = lmn$, for all $l, m, n \in \mathbf{N}$.

Clearly the sequence $(\Delta^3 x_{lmn}) \in \ell_\infty^3$.

We consider a rearrange sequence (y_{lmn}) of (x_{lmn}) defined by

$$y_{lmn} = \begin{cases} m+1, & for \quad m = l, \quad n \quad is \quad even \\ m-1, & for \quad m = l+1, \quad n \quad is \quad even \\ m, & otherwise. \end{cases}$$

Then the sequence $(\Delta^3 y_{lmn}) \notin \ell_\infty^3$.

Hence $\ell_\infty^3(\Delta^3)$ are not symmetric.

Result 3.3. The classes of sequences $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$, $\ell_\infty^3(\Delta^3)$ and $c^{3B}(\Delta^3)$ are not solid in general.

Proof. The proof is clear from the following examples:

Example 3.6. We consider the sequence (x_{lmn}) defined by

$x_{lmn} = -3$, for all $l, m, n \in \mathbf{N}$.

Clearly the difference triple sequence $(\Delta^3 x_{lmn}) \in c_0^3, c^3, c^{3R}$ and c^{3B}.

Consider the sequence of scalars defined by $\alpha_{lmn} = (-1)^{l+m+n}$ for all $l, m, n \in \mathbf{N}$.

Then the sequence $(\alpha_{lmn} x_{lmn})$ takes the following form

$\alpha_{lmn} x_{lmn} = -3.(-1)^{l+m+n}$ for all $l, m, n \in \mathbf{N}$.

Clearly $(\Delta^3 \alpha_{lmn} x_{lmn}) \notin c_0^3, c^3, c^{3R}$ and c^{3B}.

Hence $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$ and $c^{3B}(\Delta^3)$ are not solid.

Example 3.7. We consider the sequence (x_{lmn}) defined by

$$x_{lmn} = \begin{cases} lmn, & when \quad n \quad is \quad odd \qquad for \ all \ l, m \in \mathbf{N} \\ l^2 mn, & otherwise. \end{cases}$$

Clearly the sequence $(\Delta^3 x_{lmn}) \in \ell_\infty^3$

Consider the sequence of scalars defined by $\alpha_{lmn} = (-1)^{l+m}$ for all $l, m \in \mathbf{N}$.

Then the sequence $(\alpha_{lmn} x_{lmn})$ takes the following form

$$\alpha_{lmn} x_{lmn} = \begin{cases} (-1)^{l+m} lmn, & when \quad n \quad is \quad odd \qquad for \ all \ l, m \in \mathbf{N} \\ (-1)^{l+m} l^2 mn, & otherwise. \end{cases}$$

Clearly $(\Delta^3 \alpha_{lmn} x_{lmn}) \notin \ell_\infty^3$.

Hence $\ell_\infty^3(\Delta^3)$ are not solid.

Result 3.4. The classes of sequences $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$, $\ell_\infty^3(\Delta^3)$ and $c^{3B}(\Delta^3)$ are not convergence free in general.

Proof. We provide an example to prove the result:

Example 3.8. Consider the sequence defined by

$$x_{lmn} = \begin{cases} 0, & if \quad n = 1, \qquad for\ all\ l, m \in \mathbf{N} \\ 2, & otherwise. \end{cases}$$

Clearly the triple sequence $(\Delta^3 x_{lmn}) \in c_0^3, c^3, c^{3R}, \ell_\infty^3$ and c^{3B}.

Let the sequence (y_{lmn}) be defined by

$$y_{lmn} = \begin{cases} 0, & if \quad n \quad is \quad odd \qquad for\ all\ l, m \in \mathbf{N} \\ lmn, & otherwise. \end{cases}$$

Clearly $(\Delta^3 y_{lmn}) \notin c_0^3, c^3, c^{3R}, \ell_\infty^3$ and c^{3B}.

Hence $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$, $\ell_\infty^3(\Delta^3)$ and $c^{3B}(\Delta^3)$ are not convergence free.

Theorem 3.3. The classes of sequences $c_0^3(\Delta^3)$, $c^3(\Delta^3)$, $c^{3R}(\Delta^3)$, $\ell_\infty^3(\Delta^3)$ and $c^{3B}(\Delta^3)$ all are sequence algebra.

Proof. It is obvious.

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