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Multiset ideal topological spaces and local functions

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Abstract

In this article we have introduced the notion of multiset local function on an ideal topological space using the the concept of q -neighbourhood in a multiset topological space. Some basic properties of local functions on multisets have been investigated in multiset ideal topological space.

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1. Introduction

Multisets, like sets, are collection of elements but with the underlying assumption that its elements can be repeated. The number of times an element repeats is called its multiplicity. In the real world, one can speculate enormous repetition. For instance, there are many hydrogen atoms, repeated statistical datas, many strands of DNA etc. This has motivated many researcher to develop the theory of multisets which claims to have three possible relations among any two physical entity; they are different, they are same but separate, or they are coinciding and identical.

The term Multiset was first coined by N. G. deBruijn stating it as a generalization of a set. Besides its applications in the field computer science, physics, logic, many fields of mathematics have been explored in context of multisets. The topological structures of multisets have been introduced by Girish et.al. [6]. They also studied in detail some basic notions and properties like basis, limit points, closure, interior and continuity on multiset topological space Girish et.al. [7]. After the introduction of a topology on multisets many researcher have put their effort on studying many topological properties of multisets [4, 5, 6, 11].

The notion of ideal topology was introduced by Kuratowski [10]. There after Vaidyanathaswamy [20] studied the concept of ideals in point set topology. It was followed by the investigations on ideal topological and ideal fuzzy topological spaces by Tripathy and Acharjee [13], Tripathy and Ray [19]. Jankovic and Hamlett [8] investigated further properties of ideal topological spaces where they studied that ideal is a generalization, or unification, of the concept of closure point, w -accumulation point, condensation point and the point of second category. Also, on considering the ideals of subsets on N , the set of natural numbers, the notion of I -convergent sequences have been studied by Tripathy and Dutta [14], Tripathy and Hazarika ([15], [16], [17]), Tripathy and Mahanta [18] and others. The concept of local function of an ideal in general topology was further introduced by Kuratowski [10]. Zakaria et al [21] have given a brief introduction to ideals in a multiset topological space.

In this article our aim is to study the properties of local function of multiset ideals in multiset topological space.

2. Definitions and Preliminaries

In this section we procure some basic definitions, results, and notations to be used in this article.

Definition 2.1. A domain X , is defined as the set of elements from which msets are constructed. The mset space $[X]^w$ is the set of all msets whose are from X such that no element occurs more than w times.

Throughout this paper, we denote a multiset drawn from the multiset space $[X]^w$ by M .

Definition 2.2. A mset M drawn from the set X is represented by a function count M or $C_M : X \rightarrow N$ where N represents the set of non-negative integers.

Here $C_M(x)$ is the number of occurrences of the element x in the mset M drawn from the set $X = \{x_1, x_2, \dots, x_n\}$ as $M = \{m_1/x_1, m_2/x_2, \dots, m_n/x_n\}$ where m_i is the number of occurrences of the element x_i , $i = 1, 2, \dots, n$ in the mset M . The elements which are not included in the mset M have zero count, by convention.

Example 2.1. Let $X = \{p, q, r, s\}$ be any set. Then $\{3/p, 4/q, 2/r, 5/s\}$ is an mset drawn from X . Clearly a set is a particular case of an mset, in which every element has multiplicity 1.

Consider two msets M and N drawn from a set X . The following are the operations defined on the msets will be used in the article

1. $M = N$ if $C_M(x) = C_N(x)$ for all $x \in X$.
2. $M \subseteq N$ if $C_M(x) \leq C_N(x)$ for all $x \in X$.
3. $P = M \cup N$ if $C_P(x) = \max\{C_M(x), C_N(x)\}$ for all $x \in X$.
4. $P = M \cap N$ if $C_P(x) = \min\{C_M(x), C_N(x)\}$.
5. $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ for all $x \in X$.
6. $M^c = Z - M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x); x \in X\}$, Z be a mset with maximum multiplicity in the multiset space.

7. $P = M \ominus N$ if $C_P(x) = \max\{C_M(x) - C_N(x), 0\}$ for all $x \in X$,

where \oplus and \ominus represent mset addition and mset subtraction respectively.

2.1. Operations on collection of msets

Let $[X]^w$ be an mset space and Z be a subset of $[X]^w$ with $C_Z(x)$ as the multiplicities of $x \in X$ in Z . Let $\{M_1, M_2, \dots\}$ be a collection of msets drawn from $[X]^w$. Then the following operations are possible under on an arbitrary collection $\{M_i : i \in \Delta\}$ of msets, where $\Delta = \{1, 2, \dots\}$ is the support set.

1. The Union

$$\prod_{i \in \Delta} M_i = \{C_{M_i}(x)/x : C_{M_i}(x) = \max\{C_{M_i}(x) : x \in X\}\}.$$

2. The intersection

$$\cap_{i \in \Delta} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = \min\{C_{M_i}(x) : x \in X\}\}.$$

3. The mset complement

$$M^c = Z \ominus M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x), x \in X\}.$$

Definition 2.3. Let M be an mset drawn from a set X . The Support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$.

Definition 2.4. An mset M is said to be an empty set if for all $x \in X$, $C_M(x) = 0$.

Definition 2.5. Let X be a support set and $[X]^w$ be the mset space defined over X . Then for any mset $M \in [X]^w$, the complement M^c of M in $[X]^w$ is an element of $[X]^w$ such that $C_{M^c} = w - C_M(x)$ for all $x \in X$.

Since Cantor's power set theorem fails for msets, it is possible to formulate the following reasonable definition of a power mset of M for finite mset M that preserves Cantor's power set theorem.

Definition 2.6. Let $M \in [X]^w$ be an mset. The power mset $P(M)$ of M is the set of all submsets of M . We have $N \in P(M)$ if and only if $N \subseteq M$. If $N = \emptyset$, then $N \in^1 P(M)$ and if $N \neq \emptyset$, then $N \in^k P(M)$, where $k = \prod_z \binom{[M]_z}{[N]_z}$, the product \prod_z is taken over by distinct elements z

of the mset N and $|[M]_z| = m$ if and only if $z \in^m M$ and $|[N]_z| = n$ if and only if $z \in^n N$ then,

$$\binom{[M]_z}{[N]_z} = \binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$.

Note 2.1. Power mset is an mset but its support set is an ordinary set whose elements are msets.

2.2. Multiset Topology

Girish and John [6] introduced the concept of Multiset topology and defined as follows:

Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology of M if τ satisfies the following properties.

1. The mset M and the empty mset \emptyset are in τ .
2. The mset union of elements of any subcollection of τ is in τ .
3. The mset intersection of the elements of any finite subcollection of τ is in τ .

Mathematically a multiset topological space is an ordered pair (M, τ) consisting of an mset M extracted from $[X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M . Multiset topology is abbreviated as M -Topology. The elements of τ are called open mset. The complement of an open mset in a M -Topological space is said to be closed mset.

Definition 2.7. Given a subset A of M -topological space M in $[X]^w$, the *interior* of A is denoted by $int(A)$ and is defined as the mset union of all open msets contained in A .

i.e $C_{int(A)}(x) = max\{C_G(x) : G \subseteq A\}$.

Definition 2.8. Given a subset A of an M -topological space M in $[X]^w$, the *closure* of A is defined by the mset intersection of all closed msets containing A and is denoted by $cl(A)$, i.e $C_{Cl(A)}(x) = \min\{C_K(x) : A \subseteq K\}$.

The notion of ideal is defined as follows,

Definition 2.9. A nonempty collection of subsets I of a nonempty set X is called a *ideal* on X if the following postulates are satisfied

- (i) if $A \in I$ and $B \subseteq A$, then $B \in I$ (heredity),
- (ii) if $A, B \in I$, then $A \cup B \in I$ (additivity).

The triplet (X, τ, I) is called ideal fuzzy topological space with the ideal I and fuzzy topology τ .

The notion of mset ideal is defined in Zakaria et.al.[21] as follows,

Definition 2.10. A non-empty collection I of subsets of a non-empty mset M is said to be an mset ideal on M , if it satisfies the following conditions:

1. $N_1 \in I$ and $C_{N_2}(x) \leq C_{N_1}(x)$ for all $x \in X \Rightarrow N_2 \in I$.
2. $N_1 \in I, N_2 \in I \Rightarrow N_1 \cup N_2 \in I$.

The mset ideal is abbreviated as M -ideal.

Let M be an infinite set. Then consider the family of subsets by M , defined by $I = \{N \subseteq M : N \text{ is finite}\}$. It can be easily verified that I is a mset ideal of M . For further examples of mset ideals, one may refer to Zakaria et.al.[21].

It is proved by Zakaria et.al.[21] that the union and intersection of two mset ideals is a mset ideal.

3. Main Results

Definition 3.1. Let $[X]^w$ be a space of multisets. A multipoint is a multiset M in X such that

$$C_M(x) = k, \text{ for } x \in M;$$

$$= 0, \text{ otherwise,}$$

where $k \in \{1, 2, 3, \dots, w\}$ and $C_M(x)$ is the multiplicity of x in X .

Note 3.1. A multipoint, denoted by $\{k/x\}$ is a subset of a multiset M or $\{k/x\} \in M$ if $k \leq C_M(x)$.

3.1. Quasi-coincidence in Multisets

Definition 3.2. Let M be any multiset in the space $[X]^w$. If $N \subseteq M$, then $k/x \in M$ is said to be quasi-coincident with N if and only if $k > C_{N^c}(x)$.

Definition 3.3. A multiset M is said to be quasi-coincident with N i.e. MqN at x if and only if $C_M(x) > C_{N^c}$.

Remark 3.1. It can be easily verified, if M and N are quasi-coincident at x then both $C_M(x)$ and $C_N(x)$ are non-zero and hence M and N intersect at x .

Definition 3.4. A multiset N in an M -Topological space (M, τ) is said to be a q -neighborhood(Q -nbhd) of k/x if and only if there exists an open multiset P such that,

$$k/xqP \subset N.$$

3.2. Multiset Local Function

Let (M, τ) be a M -topological space and \mathcal{I} be an M -ideal on M . If N is any subset of M , then the local function denoted by $N^*(\mathcal{I}, \tau)$ is defined by,

$$N^*(\mathcal{I}, \tau)$$

$$= \cup \{m_i/x_i \in M : C_U(x_j) - C_{N^c}(x_j) > C_I(x_j), I \in \mathcal{I} \text{ for every } U \in N_q(m_i/x_i)\}$$

and at least one $x_j \in X$,
 where $N_q(m_i/x_i)$ is the set of q -nbhd of m_i/x_i .

We will write $N^*(\mathcal{I})$ or N^* in place of $N^*(\mathcal{I}, \tau)$.

Lemma 3.1. Show that $N^*(\emptyset) = ClN$.

Proof. On putting $\mathcal{I} = \emptyset$ in the expression for $N^*(I)$, we have by definition of closure in M -topology,

$$\begin{aligned} N^*(\emptyset) &= \cup\{m_i/x_i \in M : C_U(x_j) - C_{N^c}(x_j) > 0, I \in \mathcal{I}, \text{ for all } U \in N_q(m_i/x_i), \text{ and at least one } x_j \in X\} \\ &= \cup\{m_i/x_i \in M : C_U(x_j) > C_{N^c}(x_j), I \in \mathcal{I}, \text{ for all } U \in N_q(m_i/x_i), \text{ and at least one } x_j \in X\} \\ &= Cl(N). \end{aligned}$$

Example 3.2. $N^*(P^*(M)) = \emptyset$.

It is obvious, as for every $m_i/x_i \in M$ there is at least one $U \in N_q(m_i/x_i)$ such that for every $x_j \in X$, $C_U(x_j) - C_{N^c}(x_j) \leq C_I(x_j)$, $I \in P^*(M)$.

3.3. Properties of multiset local function

Throughout this section (M, τ) will denote a M -topological space with \mathcal{I} and \mathcal{J} as two M -ideals on M .

Theorem 3.1. If $N, P \subseteq M$ then, $N \subseteq P \Rightarrow N^*(\mathcal{I}, \tau) \subseteq P^*(\mathcal{I}, \tau)$.

Proof. By definition of local function, we have

$$N^*(\mathcal{I}) = \cup\{m_i/x_i \in M : C_U(x_j) - C_{N^c}(x_j) > C_I(x_j), I \in \mathcal{I} \text{ for every } U \in N_q(m_i/x_i) \text{ and at least one } x_j \in X\}.$$

$$P^*(\mathcal{I}) = \cup\{m_i/x_i \in M : C_U(x_j) - C_{P^c}(x_j) > C_I(x_j), I \in \mathcal{I} \text{ for every } U \in N_q(m_i/x_i) \text{ and at least one } x_j \in X\},$$

where $N_q(m_i/x_i)$ is the set of q -nbhd of m_i/x_i .

Since,

$$N \subseteq P$$

$$\Rightarrow C_N(x_j) \leq C_P(x_j) \forall x_j \in X$$

$$\Rightarrow C_{P^c}(x_j) \leq C_{N^c}(x_j) \text{ for all } x_j \in X$$

$$\Rightarrow C_U(x_j) - C_{N^c}(x_j) \leq C_U(x_j) - C_{P^c}(x_j).$$

Hence, $N^*(\mathcal{I}, \tau) \subseteq P^*(\mathcal{I}, \tau)$.

Theorem 3.2. If $\mathcal{I} \subseteq \mathcal{J} \Rightarrow N^*(\mathcal{J}, \tau) \subseteq N^*(\mathcal{I}, \tau)$.

Proof. We have,

$N^*(\mathcal{I}) = \cup\{m_i/x_i \in M : C_U(x_j) - C_{N^c}(x_j) > C_I(x_j), I \in \mathcal{I} \text{ for every } U \in N_q(m_i/x_i) \text{ and at least one } x_j \in X\}$.

$N^*(\mathcal{J}) = \cup\{m_i/x_i \in M : C_U(x_j) - C_{N^c}(x_j) > C_J(x_j), J \in \mathcal{J} \text{ for every } U \in N_q(m_i/x_i) \text{ and at least one } x_j \in X\}$.

Since, $\mathcal{I} \subseteq \mathcal{J}$, then there exist msets in \mathcal{J} so that for any m -point $m_i/x_i, m_i/x_i \in N^*(\mathcal{I})$ but m_i/x_i may not belong to $N^*(\mathcal{J}, \tau)$.

Hence, $N^*(\mathcal{J}, \tau) \subseteq N^*(\mathcal{I}, \tau)$.

Theorem 3.3. $(N \cup P)^*(\mathcal{I}, \tau) = N^*(\mathcal{I}, \tau) \cup P^*(\mathcal{I}, \tau)$.

Proof. Let $m/x \notin N^* \cup P^*$. Then,

$m > C_{N^* \cup P^*}(x) = \max\{C_{N^*}(x), C_{P^*}(x)\}$ for all $x \in X$, i.e $m/x \notin N^*$ and $m/x \notin P^*$.

So, for every $y \in X$ there exists at least one q -nbhd U of m/x such that

$$C_V(y) - C_P^c(y) \leq C_J(y) \text{ for some } J \in \mathcal{J}.$$

Also, for every $y \in X$, there exists at least one q -nbhd V of m/x such that

$$C_U(y) - C_N^c(y) \leq C_I(y) \text{ for some } I \in \mathcal{I}.$$

Taking, $C_W(y) = \min\{C_U(y), C_V(y)\}$, we have

$$C_W(y) - C_{N^c \cap P^c}(y) \leq C_{I \cup J}(y), I \cup J \in \mathcal{I}, \text{ i.e. } C_W(y) - C_{(N \cup P)^c} \leq C_{I \cup J}(y).$$

So, $m/x \notin (N \cup P)^*$ and hence $(N \cup P)^* \subseteq N^* \cup P^*$.

Clearly, $N^* \cup P^* \subseteq (N \cup P)^*$.

Therefore,

$$(N \cup P)^*(\mathcal{I}, \tau) = N^*(\mathcal{I}, \tau) \cup P^*(\mathcal{I}, \tau).$$

Theorem 3.4. $N^* = Cl(N^*) \subseteq Cl(N)$.

Proof. $N^*(\mathcal{I}) = \cup\{m_i/x_i \in M : C_U(x_j) - C_{N^c}(x_j) > C_I(x_j), I \in \mathcal{I} \text{ for every } U \in N_q(m_i/x_i) \text{ and at least one } x_j \in X\}$.

We have for any M -ideal \mathcal{I} , $\{\emptyset\} \subseteq \mathcal{I}$. So, by Definition 3.1 we have $N^*(\mathcal{I}) \subseteq N^*(\emptyset) = ClN$.

Therefore,

$$(3.1) \quad N^*(\mathcal{I}) \subseteq ClN.$$

Let $m_1/x_1 \in Cl(N^*)$.

So, there exists a q -nbhd U of m_1/x_1 such that UqN^* i.e $C_U(y) > C_{(N^*)^c} \forall y \in X$.

Hence $C_{N^*}(y) \neq 0$ and $C_N^*(y) = m_2$ (say).

i.e $m_2/y \in N^*$. So there exists a q -nbhd V of m_2 such that for at least one $x_2 \in X$,

$$C_V(x_2) - C_{N^c}(x_2) > C_I(x_2), I \in \mathcal{I}$$

and,

$$C_U(x_3) - C_{N^c}(x_3) > C_I(x_3), I \in \mathcal{I} \text{ for at least one } x_3 \in X$$

Since, U is a q -nbhd of m_1/x_1 , $m_1/x_1 \in N^*$.

Therefore, $Cl(N^*) \subseteq N^*$.

So, $Cl(N^*) = N^* \subseteq ClN$.

Theorem 3.5. $(N^*)^* \subseteq N^*$.

Proof. By (1), we have $(N^*)^* = Cl((N^*)^*) \subseteq Cl(N^*) = N^*$.

Theorem 3.6. $I \in \mathcal{I} \Rightarrow (N \cup I)^* = N^*$.

Proof. It is obvious that for $I \in \mathcal{I}, I^* = \emptyset$. Hence, by Theorem 3.2 we have,

$$(N \cup I)^* = N^* \cup I^* = N^*, \text{ by (2).}$$

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