Dual third-order Jacobsthal quaternions

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Abstract

In 2016, Yüce and Torunbalcı Aydın [18] defined dual Fibonacci quaternions. In this paper, we defined the dual third-order Jacobsthal quaternions and dual third-order Jacobsthal-Lucas quaternions. Also, we investigated the relations between the dual third-order Jacobsthal quaternions and third-order Jacobsthal numbers. Furthermore, we gave some their quadratic properties, the summations, the Binet’s formulas and Cassini-like identities for these quaternions.

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1. Introduction

The real quaternions are a number system which extends to the complex numbers. They were first described by Irish mathematician William Rowan Hamilton in 1843. In 1963, Horadam [9] defined the $n$-th Fibonacci quaternion which can be represented as

\[ Q_F = \{ Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} : F_n \text{ is } n\text{-th Fibonacci number} \}, \]

(1.1)

where $i^2 = j^2 = k^2 = ijk = -1$.


In 2006, Majernik [16] defined a new type of quaternions, the so-called dual quaternions in the form

\[ Q_N = \{ a + bi + cj + dk : a, b, c, d \in \mathbb{R} \}, \]

(1.2)

where $i^2 = j^2 = k^2 = ijk = -1$ and $bF_n$ is the $n$-th dual Fibonacci number.

In 2016, Yüce and Torunbalç Aydin [18] defined dual Fibonacci quaternions as follows:

\[ Q_D = \{ A + Bi + Cj + Dk : A, B, C, D \in \mathbb{D}, i^2 = j^2 = k^2 = ijk = -1 \}, \]

(1.3)

where $\mathbb{D} = \mathbb{R}[\varepsilon] = \{ a + b\varepsilon : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0 \}$. It is clear that $Q_N$ and $Q_D$ are different sets. In 2014, Nurkan and Güven [17] defined dual Fibonacci quaternions as follows:

\[ D_F = \{ Q_n = \hat{F}_n + i\hat{F}_{n+1} + j\hat{F}_{n+2} + k\hat{F}_{n+3} : \hat{F}_n = F_n + \varepsilon F_{n+1} \}, \]

(1.4)

where $i^2 = j^2 = k^2 = ijk = -1$ and $\hat{F}_n$ is the $n$-th dual Fibonacci number.


\[ \mathbf{N}_F = \{ Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3} : \]

\[ F_n \text{ is } n\text{-th Fibonacci number}, \]

(1.5)

where \( i^2 = j^2 = k^2 = 0, \ ij = -ji = jk = -kj = ki = -ik = 0 \). For more details on dual quaternions and generalized dual Fibonacci quaternions, see [5, 19].

On the other hand, the Jacobsthal numbers have many interesting properties and applications in many fields of science (see, e.g., [2]). The Jacobsthal numbers \( J_n \) are defined by the recurrence relation

(1.6)

\[ J_0 = 0, \ J_1 = 1, \ J_{n+1} = J_n + 2J_{n-1}, \ n \geq 1. \]

Another important sequence is the Jacobsthal-Lucas sequence. This sequence is defined by the recurrence relation \( j_{n+1} = j_n + 2j_{n-1}, \ n \geq 1 \) and \( j_0 = 2, \ j_1 = 1 \). (see, [11]).

In [4], the Jacobsthal recurrence relation (1.6) is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [11] is expanded and extended to several identities for some of the higher order cases. In particular, third-order Jacobsthal numbers, \( \{ J_n^{(3)} \}_{n \geq 0} \), and third-order Jacobsthal-Lucas numbers, \( \{ j_n^{(3)} \}_{n \geq 0} \), are defined by

(1.7) \[ J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)}, \ J_0^{(3)} = 0, \ J_1^{(3)} = J_2^{(3)} = 1, \ n \geq 0, \]

and

(1.8) \[ j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)}, \ j_0^{(3)} = 2, \ j_1^{(3)} = 1, \ j_2^{(3)} = 5, \ n \geq 0, \]

respectively.

The following properties given for third order Jacobsthal numbers and third order Jacobsthal-Lucas numbers play important roles in this paper (for more, see [3, 4]).

(1.9) \[ 3J_n^{(3)} + j_n^{(3)} = 2^{n+1}, \]
\begin{align}
(1.10) \quad j^{(3)}_n - 3J^{(3)}_n &= 2j^{(3)}_{n-3}, \\
(1.11) \quad J^{(3)}_{n+2} - 4J^{(3)}_n &= \begin{cases}
-2 & \text{if } n \equiv 1 \pmod{3} \\
1 & \text{if } n \not\equiv 1 \pmod{3}
\end{cases}, \\
(1.12) \quad j^{(3)}_n - 4J^{(3)}_n &= \begin{cases}
2 & \text{if } n \equiv 0 \pmod{3} \\
-3 & \text{if } n \equiv 1 \pmod{3} \\
1 & \text{if } n \equiv 2 \pmod{3}
\end{cases}, \\
(1.13) \quad j^{(3)}_{n+1} + j^{(3)}_n &= 3J^{(3)}_{n+2}, \\
(1.14) \quad j^{(3)}_n - J^{(3)}_{n+2} &= \begin{cases}
1 & \text{if } n \equiv 0 \pmod{3} \\
-1 & \text{if } n \equiv 1 \pmod{3} \\
0 & \text{if } n \equiv 2 \pmod{3}
\end{cases}, \\
(1.15) \quad \left( j^{(3)}_{n-3} \right)^2 + 3J^{(3)}_n j^{(3)}_n &= 4^n, \\
(1.16) \quad \sum_{k=0}^n J^{(3)}_k &= \begin{cases}
J^{(3)}_{n+1} & \text{if } n \not\equiv 0 \pmod{3} \\
J^{(3)}_{n+1} - 1 & \text{if } n \equiv 0 \pmod{3}
\end{cases}
\text{ and } \\
(1.17) \quad \left( j^{(3)}_n \right)^2 - 9 \left( J^{(3)}_n \right)^2 &= 2^{n+2}j^{(3)}_{n-3}.
\end{align}

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

\[ x^3 - x^2 - x - 2 = 0; \quad x = 2, \quad \text{and} \quad x = \frac{-1 \pm i\sqrt{3}}{2}. \]

Note that the latter two are the complex conjugate cube roots of unity. Call them \( \omega_1 \) and \( \omega_2 \), respectively. Thus the Binet formulas can be written as

\begin{align}
(1.18) \quad J^{(3)}_n &= \frac{1}{7}2^{n+1} - \frac{3}{21} + \frac{2i\sqrt{3}}{21}\omega_1^n - \frac{3 - 2i\sqrt{3}}{21}\omega_2^n = \frac{1}{7} \left( 2^{n+1} - V^{(3)}_n \right),
\end{align}

and
\[ j^{(3)}_n = \frac{1}{7} 2^{n+3} + \frac{3 + 2i\sqrt{3}}{7} \omega_1^n + \frac{3 - 2i\sqrt{3}}{7} \omega_2^n = \frac{1}{7} \left( 2^{n+3} + 3V^{(3)}_n \right), \]

respectively. Here \( V^{(3)}_n \) is the sequence defined by

\[ V^{(3)}_n = \frac{2 + 2i\sqrt{3}}{3} \omega_1^n + \frac{3 - 2i\sqrt{3}}{3} \omega_2^n = \begin{cases} 2 & \text{if } n \equiv 0 \mod 3 \\ -3 & \text{if } n \equiv 1 \mod 3 \\ 1 & \text{if } n \equiv 2 \mod 3 \end{cases}. \]

Using Eq. (1.20) is easy to see that for all \( n \geq 0 \):

\[ V^{(3)}_n + 2V^{(3)}_{n+1} + 4V^{(3)}_{n+2} = \begin{cases} 0 & \text{if } n \equiv 0 \mod 3 \\ 7 & \text{if } n \equiv 1 \mod 3 \\ -7 & \text{if } n \equiv 2 \mod 3 \end{cases}. \]

Recently in [3], we have defined a new type of quaternions with the third-order Jacobsthal and third-order Jacobsthal-Lucas number components as

\[ JQ^{(3)}_n = J^{(3)}_n + J^{(3)}_{n+1}i + J^{(3)}_{n+2}j + J^{(3)}_{n+3}k \]

and

\[ jQ^{(3)}_n = j^{(3)}_n + j^{(3)}_{n+1}i + j^{(3)}_{n+2}j + j^{(3)}_{n+3}k, \]

respectively, where \( i^2 = j^2 = k^2 = ijk = -1 \), and we studied the properties of these quaternions. Also, we derived the generating functions and many other identities for the third-order Jacobsthal and third-order Jacobsthal-Lucas quaternions.

In this paper, we define the dual third-order Jacobsthal quaternions and dual third-order Jacobsthal-Lucas quaternions as follows:

\[ (1.21) \quad JN^{(3)}_m = J^{(3)}_m + J^{(3)}_{m+1}i + J^{(3)}_{m+2}j + J^{(3)}_{m+3}k \quad (m \geq 0) \]

and

\[ (1.22) \quad jN^{(3)}_m = j^{(3)}_m + j^{(3)}_{m+1}i + j^{(3)}_{m+2}j + j^{(3)}_{m+3}k \quad (m \geq 0), \]
respectively. Here \( i^2 = j^2 = k^2 = 0 \), \( ij = -ji = jk = -kj = ki = -ik = 0 \). Also, we investigated the relations between the dual third-order Jacobsthal quaternions and third-order Jacobsthal numbers. Furthermore, we give some their quadratic properties, the Binet’s formulas, d’Ocagne and Cassini-like identities for these quaternions.

2. Dual Third-Order Jacobsthal Quaternions

We can define dual third-order Jacobsthal quaternions by using third-order Jacobsthal numbers. The \( n \)-th third-order Jacobsthal number \( J^{(3)}_n \) is defined by Eq. (1.7). Then, we can define the dual third-order Jacobsthal quaternions as follows:

\[
\mathbf{N}_J = \{ J^{(3)}_m = J^{(3)}_m + J^{(3)}_{m+1} i + J^{(3)}_{m+2} j + J^{(3)}_{m+3} k : m \geq 0 \},
\]

where \( J^{(3)}_m \) is the \( m \)-th third-order Jacobsthal number and \( \{i, j, k\} \) as in Eq. (1.2). Also, we can define the dual third-order Jacobsthal-Lucas quaternion as follows:

\[
\mathbf{N}_j = \{ j^{(3)}_m = j^{(3)}_m + j^{(3)}_{m+1} i + j^{(3)}_{m+2} j + j^{(3)}_{m+3} k : m \geq 0 \},
\]

where \( j^{(3)}_m \) is the \( m \)-th third-order Jacobsthal-Lucas number.

Then, the addition and subtraction of the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions is defined by

\[
J^{(3)}_m \pm j^{(3)}_m = (J^{(3)}_m + j^{(3)}_{m+1} i + j^{(3)}_{m+2} j + j^{(3)}_{m+3} k) \\
\pm (j^{(3)}_m + j^{(3)}_{m+1} i + j^{(3)}_{m+2} j + j^{(3)}_{m+3} k) \\
= (J^{(3)}_m \pm j^{(3)}_m) + (j^{(3)}_{m+1} \pm j^{(3)}_{m+1}) i + (j^{(3)}_{m+2} \pm j^{(3)}_{m+2}) j \\
+(j^{(3)}_{m+3} \pm j^{(3)}_{m+3}) k
\]

and the multiplication of the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions is defined by
\[ JN_m^{(3)} jN_m^{(3)} \]

\[(2.4) \quad (J_m^{(3)} + J_{m+1}^{(3)} i + J_{m+2}^{(3)} j + J_{m+3}^{(3)} k)(J_m^{(3)} + J_{m+1}^{(3)} i + J_{m+2}^{(3)} j + J_{m+3}^{(3)} k) = J_m^{(3)} jN_m^{(3)} + (J_m^{(3)} j_{m+1}^{(3)} + J_m^{(3)} j_{m+2}^{(3)} + J_m^{(3)} j_{m+3}^{(3)}) j + (J_m^{(3)} j_{m+3}^{(3)} + J_{m+3}^{(3)} j_m^{(3)}) k. \]

Now, the scalar and the vector part of the \( JN_m^{(3)} \) which is the \( m \)-th term of the dual third-order Jacobsthal sequence \( \{JN_m^{(3)}\}_{m \geq 0} \) are denoted by \( S_{JN_m^{(3)}} \) and \( V_{JN_m^{(3)}} \), respectively.

Thus, the dual third-order Jacobsthal \( JN_m^{(3)} \) is given by \( S_{JN_m^{(3)}} + V_{JN_m^{(3)}} \).

Then, relation (2.4) is defined by

\[(2.6) \quad JN_m^{(3)} jN_m^{(3)} = S_{JN_m^{(3)}} S_{J_{N_m^{(3)}}} + S_{JN_m^{(3)}} V_{J_{N_m^{(3)}}} + S_{J_{N_m^{(3)}}} V_{JN_m^{(3)}}. \]

The conjugate of dual third-order Jacobsthal quaternion \( JN_m^{(3)} \) is denoted by \( \overline{JN_m^{(3)}} \) and it is \( \overline{JN_m^{(3)}} = J_m^{(3)} - J_{m+1}^{(3)} i - J_{m+2}^{(3)} j - J_{m+3}^{(3)} k \). The norm of \( JN_m^{(3)} \) is defined as

\[(2.7) \quad N_{r_2}^2(JN_m^{(3)}) = JN_m^{(3)} \overline{JN_m^{(3)}} = JN_m^{(3)} JN_m^{(3)} = (JN_m^{(3)})^2. \]

Then, we give the following theorem using statements (2.1), (2.3) and (2.4).

**Theorem 2.1.** Let \( J_m^{(3)} \) and \( JN_m^{(3)} \) be the \( m \)-th terms of the third-order Jacobsthal sequence \( \{J_m^{(3)}\}_{m \geq 0} \) and the dual third-order Jacobsthal quaternion sequence \( \{JN_m^{(3)}\}_{m \geq 0} \), respectively. In this case, for \( m \geq 0 \) we can give the following relations:

\[(2.8) \quad 2JN_m^{(3)} + JN_{m+1}^{(3)} + JN_{m+2}^{(3)} = JN_{m+3}^{(3)}, \]

\[(2.9) \quad JN_m^{(3)} - JN_{m+1}^{(3)} i - JN_{m+2}^{(3)} j - JN_{m+3}^{(3)} k = J_m^{(3)}. \]
\[
\left( JN_{m}^{(3)} \right)^2 + \left( JN_{m+1}^{(3)} \right)^2 + \left( JN_{m+2}^{(3)} \right)^2 = \frac{1}{7} \begin{pmatrix}
3 \cdot 2^{2(m+1)}(1 + 4i + 8j + 16k) \\
-2^{m+2}U_{m}^{(3)} \\
-2^{m+3}U_{m}^{(3)}(i + 2j + 4k) \\
+2(1 - i - j + 2k)
\end{pmatrix},
\]

(2.10)

where

\[
U_{m}^{(3)} = U_{m}^{(3)} + U_{m+1}^{(3)}i + U_{m+2}^{(3)}j + U_{m+3}^{(3)}k \quad \text{and} \quad U_{m}^{(3)} = \frac{1}{7} \left( V_{m+1}^{(3)} + 3V_{m+2}^{(3)} \right).
\]

Proof. (2.8): By the equations

\[
JN_{m}^{(3)} = J_{m}^{(3)} + J_{m+1}^{(3)}i + J_{m+2}^{(3)}j + J_{m+3}^{(3)}k
\]

and (1.7), we get

\[
2JN_{m}^{(3)} + JN_{m+1}^{(3)} + JN_{m+2}^{(3)} = (2J_{m}^{(3)} + 2J_{m+1}^{(3)}i + 2J_{m+2}^{(3)}j + 2J_{m+3}^{(3)}k) + (J_{m+1}^{(3)} + J_{m+2}^{(3)}i + J_{m+3}^{(3)}j + J_{m+4}^{(3)}k) + (J_{m+2}^{(3)} + J_{m+3}^{(3)}i + J_{m+4}^{(3)}j + J_{m+5}^{(3)}k) = (2J_{m+1}^{(3)} + J_{m+2}^{(3)}j + 2J_{m+1}^{(3)}j + J_{m+3}^{(3)}j) + (2J_{m+3}^{(3)} + J_{m+4}^{(3)}j + 2J_{m+3}^{(3)}j + J_{m+5}^{(3)}j) = J_{m+3}^{(3)} + J_{m+4}^{(3)}j + J_{m+5}^{(3)}k
\]

= \frac{1}{7} \left( V_{m+1}^{(3)} + 3V_{m+2}^{(3)} \right).
\]

(2.9): By using \(JN_{m}^{(3)}\) in the Eq. (2.1) and conditions (1.2), we get

\[
JN_{m}^{(3)} - JN_{m+1}^{(3)}i - JN_{m+2}^{(3)}j - JN_{m+3}^{(3)}k = J_{m}^{(3)} + J_{m+1}^{(3)}i + J_{m+2}^{(3)}j + J_{m+3}^{(3)}k - (J_{m+1}^{(3)} + J_{m+2}^{(3)}i + J_{m+3}^{(3)}j) - (J_{m+2}^{(3)} + J_{m+3}^{(3)}i + J_{m+4}^{(3)}j) - (J_{m+3}^{(3)} + J_{m+4}^{(3)}i + J_{m+5}^{(3)}j) = J_{m}^{(3)}.
\]

(2.10): By using Eqs. (2.4) and (1.18), we get

\[
(2.11) \quad \left( JN_{m}^{(3)} \right)^2 = \left( J_{m}^{(3)} \right)^2 + 2J_{m}^{(3)}J_{m+1}^{(3)}i + 2J_{m}^{(3)}J_{m+2}^{(3)}j + 2J_{m}^{(3)}J_{m+3}^{(3)}k
\]

and
\[
\begin{align*}
\left( J_m^{(3)} \right)^2 &+ \left( J_{m+1}^{(3)} \right)^2 + \left( J_{m+2}^{(3)} \right)^2 \\
(2.12) &= \frac{1}{39} \left( (2^{m+3} - V_{m+1}^{(3)})^2 + (2^{m+2} - V_{m+1}^{(3)})^2 + (2^{m+3} - V_{m+2}^{(3)})^2 \right) \\
&= \frac{1}{39} \left( 21 \cdot 2^{2(m+1)} - 2^{m+2}(V_m^{(3)} + 2V_{m+1}^{(3)} + 4V_{m+2}^{(3)} + 14) \right) \\
&= \frac{1}{7} \left( 3 \cdot 2^{2(m+1)} - 2^{m+2}U_m^{(3)} + 2 \right),
\end{align*}
\]

where \( U_m^{(3)} = \frac{1}{7} \left( V_m^{(3)} + 2V_{m+1}^{(3)} + 4V_{m+2}^{(3)} \right) = \frac{1}{7} \left( V_{m+1}^{(3)} + 3V_{m+2}^{(3)} \right) \).

Finally, from the Eqs. (2.11) and (2.12), we obtain
\[
\begin{align*}
\left( J_N^{(3)} \right)^2 &+ \left( J_{N+1}^{(3)} \right)^2 + \left( J_{N+2}^{(3)} \right)^2 \\
(2.12) &= \frac{1}{39} \left( (2^{m+3} - V_{m+1}^{(3)})^2 + (2^{m+2} - V_{m+1}^{(3)})^2 + (2^{m+3} - V_{m+2}^{(3)})^2 \right) \\
&= \frac{1}{7} \left( 3 \cdot 2^{2(m+1)} - 2^{m+2}U_{N+1}^{(3)} + 2 \right),
\end{align*}
\]

where \( U_{N+1}^{(3)} = U_m^{(3)} + U_{m+1}^{(3)} + U_{m+2}^{(3)} + U_{m+3}^{(3)} \).

Theorem 2.2. Let \( J_m^{(3)} \) and \( j_m^{(3)} \) be the \( m \)-th terms of the dual third-order Jacobsthal quaternion sequence \( \{ J_m^{(3)} \} \) and the dual third-order Jacobsthal-Lucas quaternion sequence \( \{ j_m^{(3)} \} \), respectively. The following relations are satisfied
\[
\begin{align*}
(2.13) &\quad j_{N+3}^{(3)} - 3j_{N+3}^{(3)} = 2j_{N+3}^{(3)}, \\
(2.14) &\quad j_{N+3}^{(3)} + j_{N+3}^{(3)} = 3j_{N+3}^{(3)}, \\
(2.15) &\quad \left( j_{N+3}^{(3)} \right)^2 + 3j_{N+3}^{(3)}j_{N+3}^{(3)} = 4^{m+3}(1 + 4i + 8j + 16k).
\end{align*}
\]
Since $VN$ and third-order Jacobsthal-Lucas number (1.10) and (2.3), it follows that

$$\sum_{n=0}^{m} J_{m+3} = J_{m+3}^{(3)} + J_{m+4}^{(3)} + J_{m+5}^{(3)} + J_{m+6}^{(3)}.$$

Proof. (2.13): From identities between third-order Jacobsthal number and third-order Jacobsthal-Lucas number (1.10) and (2.3), it follows that

$$\sum_{n=0}^{m} J_{m+3} = J_{m+3}^{(3)} + J_{m+4}^{(3)} + J_{m+5}^{(3)} + J_{m+6}^{(3)}.$$

The proof of (2.14) is similar to (2.13), using the identity (1.13). (2.15):

Now, using Eqs. (2.4), (2.11) and (1.15), we get

$$\sum_{n=0}^{m} J_{m+3} = J_{m+3}^{(3)} + J_{m+4}^{(3)} + J_{m+5}^{(3)} + J_{m+6}^{(3)}.$$

The last equality because $3J_{m+3}^{(3)} = 4^{m+3} - \left( J_{m}^{(3)} \right)^2$ in Eq. (1.15). \(\Box\)

Theorem 2.3. Let $J_{m}^{(3)}$ be the $m$-th term of the dual third-order Jacobsthal quaternion sequence $\{ J_{m}^{(3)} \}_{m \geq 0}$. Then, we have the following identity

$$\sum_{s=0}^{m} J_{s}^{(3)} = J_{m+1}^{(3)} - \frac{1}{21} \left( 7(1 + 4j + 7k) - 4VN_{m+1} + VN_{m} \right),$$

(2.16)

where $VN_{m} = V_{m}^{(3)} + V_{m+1}^{(3)} + V_{m+2}^{(3)} + V_{m+3}^{(3)}.$

Proof. Since

$$\sum_{s=0}^{m} J_{s}^{(3)} = J_{m+1}^{(3)} - \frac{1}{21} \left( 7 - 4V_{m+1}^{(3)} + V_{m}^{(3)} \right) = \begin{cases} J_{m+1}^{(3)} & \text{if } n \not\equiv 0 \pmod{3} \\ J_{m+1}^{(3)} - 1 & \text{if } n \equiv 0 \pmod{3} \end{cases}$$

(2.17)
Theorem 2.4. Let $JN_m^3$ and $jN_m^3$ be the $m$-th terms of the dual third-order Jacobsthal quaternion sequence $\{JN_m^3\}_{m \geq 0}$ and the dual third-order Jacobsthal-Lucas quaternion sequence $\{jN_m^3\}_{m \geq 0}$, respectively. Then, we have

\begin{align}
(2.18) & \quad jN_m^3 \overline{JN}_m^3 - \overline{jN}_m^3 JN_m^3 = 2(J_m^3 jN_m^3 - j_m^3 JN_m^3), \\
(2.19) & \quad jN_m^3 JN_m^3 + \overline{jN}_m^3 \overline{JN}_m^3 = 2j_m^3 J_m^3.
\end{align}

Proof. (2.18): By the Eqs. (2.1), (2.2) and $\overline{JN}_m^3 = J_m^3 - J_m^3 i - J_m^3 + J_m^3 k$, we get $jN_m^3 \overline{JN}_m^3 - \overline{jN}_m^3 JN_m^3 = 2(J_m^3 jN_m^3 - j_m^3 JN_m^3)$.

(2.19): $jN_m^3 JN_m^3 + \overline{jN}_m^3 \overline{JN}_m^3 = 2j_m^3 J_m^3$. 

(see [3]), we get

\[
\begin{align}
\sum_{s=0}^{m} JN_s^3 &= \sum_{s=0}^{m} J_s^3 + i \sum_{s=1}^{m+1} J_s^3 + j \sum_{s=2}^{m+2} J_s^3 + k \sum_{s=3}^{m+3} J_s^3 \\
&= J_m^3 - \frac{1}{21} \left(7 - 4V_{m+1}^3 + V_m^3\right) \\
&+ \left(J_{m+2}^3 - \frac{1}{21} \left(7 - 4V_{m+2}^3 + V_{m+1}^3\right)\right)i \\
&+ \left(J_{m+3}^3 - \frac{1}{21} \left(28 - 4V_{m+3}^3 + V_{m+2}^3\right)\right)j \\
&+ \left(J_{m+4}^3 - \frac{1}{21} \left(49 - 4V_{m+4}^3 + V_{m+3}^3\right)\right)k \\
&= JN_{m+1}^3 - \frac{1}{21} \left(7(1 + i + 4j + 7k) - 4V_{m+1}^3 + V_{m+1}^3\right),
\end{align}
\]

where $V_{m+1}^3 = V_m^3 + V_{m+1}^3 i + V_{m+2}^3 j + V_{m+3}^3 k$. □
Theorem 2.5 (Binet’s Formulas). Let $JN^{(3)}_m$ and $jN^{(3)}_m$ be $m$-th terms of the dual third-order Jacobsthal quaternion sequence $\{JN^{(3)}_m\}_{m \geq 0}$ and the dual third-order Jacobsthal-Lucas quaternion sequence $\{jN^{(3)}_m\}_{m \geq 0}$, respectively. For $m \geq 0$, the Binet’s formulas for these quaternions are as follows:

\[
JN^{(3)}_m = \frac{1}{7} 2^{m+1} \alpha - \frac{3 + 2i\sqrt{3}}{21} \omega_1 \omega_1^m - \frac{3 - 2i\sqrt{3}}{21} \omega_2 \omega_2^m = \frac{1}{7} \left( 2^{m+1} \alpha - VN^{(3)}_m \right)
\]  
(2.20)

and

\[
jN^{(3)}_m = \frac{1}{7} 2^{m+3} \alpha + \frac{3 + 2i\sqrt{3}}{7} \omega_1 \omega_1^m - \frac{3 - 2i\sqrt{3}}{7} \omega_2 \omega_2^m = \frac{1}{7} \left( 2^{m+3} \alpha + 3VN^{(3)}_m \right),
\]  
(2.21)

respectively, where $VN^{(3)}_m$ is the sequence defined by

\[
VN^{(3)}_m = \begin{cases} 2 - 3i + j + 2k & \text{if } n \equiv 0 \pmod{3} \\ -3 + i + 2j - 3k & \text{if } n \equiv 1 \pmod{3} \\ 1 + 2i - 3j + k & \text{if } n \equiv 2 \pmod{3} \end{cases},
\]  
(2.22)

$\alpha = 1 + 2i + 4j + 8k$ and $\omega_{1,2} = 1 + \omega_{1,2}i + \omega_{1,2}^2j + k$.

Proof. Repeated use of (1.18) in (2.1) enables one to write for $\alpha = 1 + 2i + 4j + 8k$ and $\omega_{1,2} = 1 + \omega_{1,2}i + \omega_{1,2}^2j + k$,

\[
JN^{(3)}_m = J^{(3)}_m + J^{(3)}_{m+1}i + J^{(3)}_{m+2}j + J^{(3)}_{m+3}k
\]

\[
= \frac{1}{7} 2^{m+1} \alpha - \frac{3 + 2i\sqrt{3}}{21} \omega_1 \omega_1^m - \frac{3 - 2i\sqrt{3}}{21} \omega_2 \omega_2^m
\]

\[
+ \left( \frac{1}{7} 2^{m+2} - \frac{3 + 2i\sqrt{3}}{21} \omega_1 \omega_1^{m+1} - \frac{3 - 2i\sqrt{3}}{21} \omega_2 \omega_2^{m+1} \right) i
\]

\[
+ \left( \frac{1}{7} 2^{m+3} - \frac{3 + 2i\sqrt{3}}{21} \omega_1 \omega_1^{m+2} - \frac{3 - 2i\sqrt{3}}{21} \omega_2 \omega_2^{m+2} \right) j
\]

\[
+ \left( \frac{1}{7} 2^{m+4} - \frac{3 + 2i\sqrt{3}}{21} \omega_1 \omega_1^{m+3} - \frac{3 - 2i\sqrt{3}}{21} \omega_2 \omega_2^{m+3} \right) k
\]

\[
= \frac{1}{7} 2^{m+1} \alpha + \frac{3 + 2i\sqrt{3}}{7} \omega_1 \omega_1^m + \frac{3 - 2i\sqrt{3}}{7} \omega_2 \omega_2^m.
\]  
(2.23)
and similarly making use of (1.19) in (2.2) yields

\[ jN_m^{(3)} = j_m^{(3)} + j_{m+1}^{(3)} i + j_{m+2}^{(3)} j + j_{m+3}^{(3)} k = \frac{1}{7} 2^m + 3 + 2i\sqrt{3} \omega_1^m + 3 - 2i\sqrt{3} \omega_2^m. \]  
(2.24)

The formulas in (2.23) and (2.24) are called as Binet’s formulas for the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions, respectively. Using notation in (2.22), we obtain the results (2.20) and (2.21).

**Theorem 2.6 (D’Ocagne-like Identity).** Let \( JN_m^{(3)} \) be the \( m \)-th terms of the dual third-order Jacobsthal quaternion sequence \( \{ JN_m^{(3)} \}_{m \geq 0} \). In this case, for \( n \geq m \geq 0 \), the d’Ocagne identities for \( JN_m^{(3)} \) is as follows:

\[ JN_n^{(3)} JN_{m+1}^{(3)} - JN_{n+1}^{(3)} JN_m^{(3)} = \frac{1}{7} \left( \alpha \left( 2^{m+1} U N_{m+1}^{(3)} - 2^m U N_{n+1}^{(3)} \right) + (1 - i - j + 2k) U_{n-m}^{(3)} \right), \]

(2.25)

\[ \left( JN_{m+1}^{(3)} \right)^2 - JN_{m+2}^{(3)} JN_m^{(3)} = \frac{1}{7} \left( 2^{m+1} \alpha (2 U N_{m+1}^{(3)} - U N_{m+2}^{(3)}) + (1 - i - j + 2k) \right), \]

(2.26)

where \( U N_{m+1}^{(3)} = \frac{1}{7} (2 V N_m^{(3)} - V N_{m+1}^{(3)}) \), \( \alpha = 1 + 2i + 4j + 8k \) and \( U_n^{(3)} \) as in Eq. (2.12).
Proof. (2.25): Using Eqs. (2.20) and (2.22), we get

\[
JN_n^{(3)} - JN_{n+1}^{(3)} - JN_m^{(3)}
\]

\[
= \frac{1}{17} \left( \begin{pmatrix} 2^{n+1} \alpha - VN_n^{(3)} & 2^{n+2} \alpha - VN_{m+1}^{(3)} \\ 2^{n+2} \alpha - VN_{n+1}^{(3)} & 2^{m+1} \alpha - VN_m^{(3)} \end{pmatrix} \right)
\]

\[
= \frac{1}{17} \left( \begin{pmatrix} 2^{n+m+3} \alpha^2 - 2^{n+1} \alpha VN_m^{(3)} & -2^{m+2} VN_n^{(3)} \alpha + VN_n^{(3)} VN_{m+1}^{(3)} \\ -2^{n+m+3} \alpha^2 + 2^{m+2} \alpha VN_m^{(3)} + 2^{m+1} VN_n^{(3)} \alpha - VN_n^{(3)} VN_m^{(3)} \end{pmatrix} \right)
\]

\[
= \frac{1}{7} \left( \alpha \left( 2^{n+1} UN_{m+1}^{(3)} - 2^{m+1} UN_{n+1}^{(3)} \right) + (1 - i - j + 2k) U_n^{(3)} \right),
\]

(2.27)

where \( UN_m^{(3)} = \frac{1}{7}(2VN_m^{(3)} - VN_{m+1}^{(3)}) \) and \( VN_m^{(3)} \) as in (2.22). In particular, if \( n = m + 1 \) in Eq. (2.27), we obtain for \( m \geq 0, \)

\[
\left( JN_{m+1}^{(3)} \right)^2 - JN_{m+2}^{(3)} JN_m^{(3)} = \frac{1}{7} \left( 2^{m+1} \alpha \left( 2UN_{m+1}^{(3)} - UN_{m+2}^{(3)} \right) \right).
\]

(2.28)

We will give an example in which we check in a particular case the Cassini-like identity for dual third-order Jacobsthal quaternions.

Example 2.7. Let \( \{JN_s^{(3)} : s = 0, 1, 2, 3\} \) be the dual third-order Jacobsthal quaternions such that \( JN_0^{(3)} = i + j + 2k, JN_1^{(3)} = 1 + i + 2j + 5k, JN_2^{(3)} = 1 + 2i + 5j + 9k \) and \( JN_3^{(3)} = 2 + 5i + 9j + 18k \). In this case,

\[
\left( JN_1^{(3)} \right)^2 - JN_2^{(3)} JN_0^{(3)}
\]

\[
= (1 + i + 2j + 5k)^2 - (1 + 2i + 5j + 9k)(i + j + 2k)
\]

\[
= (1 + 2i + 4j + 10k) - (i + j + 2k)
\]

\[
= 1 + i + 3j + 8k
\]

\[
= \frac{1}{7} \left( 2(1 + 2i + 4j + 8k)(2UN_1^{(3)} - UN_2^{(3)}) \right)
\]

\[
+ (1 - i - j + 2k)
\]
and
\[
\left( JN_2^{(3)} \right)^2 - JN_3^{(3)} JN_1^{(3)}
= (1 + 2i + 5j + 9k)^2 - (2 + 5i + 9j + 18k)(1 + i + 2j + 5k)
= (1 + 4i + 10j + 18k) - (2 + 7i + 13j + 28k)
= -1 - 3i - 3j - 10k
= \frac{1}{7} \left( 4(1 + 2i + 4j + 8k)(2UN_2^{(3)} - UN_3^{(3)}) + (1 - i - j + 2k) \right).
\]

3. Conclusions

There are two differences between the dual third-order Jacobsthal and the dual coefficient third-order Jacobsthal quaternions. The first one is as follows: the dual coefficient third-order Jacobsthal quaternionic units are \(i^2 = j^2 = k^2 = ijk = -1\) whereas the dual third-order Jacobsthal quaternionic units are \(i^2 = j^2 = k^2 = 0,\ ij = -ji = jk = -kj = ki = -ik = 0\). The second one is as follows: the elements of the dual coefficient third-order Jacobsthal quaternion are \(J_m^{(3)} + \varepsilon J_{m+1}^{(3)} (\varepsilon^2 = 0, \ \varepsilon \neq 0)\) whereas the elements of the dual third-order Jacobsthal quaternions are \(m\)-th third-order Jacobsthal number \(J_m^{(3)}\).

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References


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