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Dual third-order Jacobsthal quaternions

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Abstract

In 2016, Yüce and Torunbalcı Aydın [18] defined dual Fibonacci quaternions. In this paper, we defined the dual third-order Jacobsthal quaternions and dual third-order Jacobsthal-Lucas quaternions. Also, we investigated the relations between the dual third-order Jacobsthal quaternions and third-order Jacobsthal numbers. Furthermore, we gave some their quadratic properties, the summations, the Binet's formulas and Cassini-like identities for these quaternions.

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1. Introduction

The real quaternions are a number system which extends to the complex numbers. They are first described by Irish mathematician William Rowan Hamilton in 1843. In 1963, Horadam [9] defined the n-th Fibonacci quaternion which can be represented as

$$Q_F = \{Q_n = F_n + \mathbf{i}F_{n+1} + \mathbf{j}F_{n+2} + \mathbf{k}F_{n+3} : \ F_n \ \text{ is } n\text{−}th \text{ Fibonacci number}\}, \tag{1.1}$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$.

In 1969, Iyer [14, 15] derived many relations for the Fibonacci quaternions. In 1977, Iakin [12, 13] introduced higher order quaternions and gave some identities for these quaternions. Furthermore, Horadam [10] extend to quaternions to the complex Fibonacci numbers defined by Harman [6]. In 2012, Halıcı [6] gave generating functions and Binet's formulas for Fibonacci and Lucas quaternions.

In 2006, Majernik [16] defined a new type of quaternions, the so-called dual quaternions in the form $Q_{\mathbf{N}} = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} : \ a, b, c, d \in \mathbf{R}\}$, with the following multiplication schema for the quaternion units

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 0, \ \mathbf{ij} = -\mathbf{ji} = \mathbf{jk} = -\mathbf{kj} = \mathbf{ki} = -\mathbf{ik} = 0. \tag{1.2}$$

In 2009, Ata and Yaylı [1] defined dual quaternions with dual numbers coefficient as follows:

$$Q_{\mathbf{D}} = \{A + B\mathbf{i} + C\mathbf{j} + D\mathbf{k} : \ A, B, C, D \in \mathbf{D}, \ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\}, \tag{1.3}$$

where $\mathbf{D} = \mathbf{R}[\varepsilon] = \{a + b\varepsilon : \ a, b \in \mathbf{R}, \ \varepsilon^2 = 0, \ \varepsilon \neq 0\}$. It is clear that $Q_{\mathbf{N}}$ and $Q_{\mathbf{D}}$ are different sets. In 2014, Nurkan and Güven [17] defined dual Fibonacci quaternions as follows:

$$\mathbf{D}_F = \{Q_n = \widehat{F}_n + \mathbf{i}\widehat{F}_{n+1} + \mathbf{j}\widehat{F}_{n+2} + \mathbf{k}\widehat{F}_{n+3} : \ \widehat{F}_n = F_n + \varepsilon F_{n+1}\}, \tag{1.4}$$

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ and $\widehat{F}_n$ is the n-th dual Fibonacci number.

In 2016, Yüce and Torunbalcı Aydın [18] defined dual Fibonacci quaternions as follows:

$$\mathbf{N}_F = \{Q_n = F_n + \mathbf{i}F_{n+1} + \mathbf{j}F_{n+2} + \mathbf{k}F_{n+3} :$$

F_n is n-th Fibonacci number},

(1.5)

where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 0,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{jk} = -\mathbf{kj} = \mathbf{ki} = -\mathbf{ik} = 0$. For more details on dual quaternions and generalized dual Fibonacci quaternions, see [5, 19].

On the other hand, the Jacobsthal numbers have many interesting properties and applications in many fields of science (see, e.g., [2]). The Jacobsthal numbers J_n are defined by the recurrence relation

$$J_0 = 0,\ J_1 = 1,\ J_{n+1} = J_n + 2J_{n-1},\ n \geq 1. \tag{1.6}$$

Another important sequence is the Jacobsthal-Lucas sequence. This sequence is defined by the recurrence relation $j_{n+1} = j_n + 2j_{n-1},\ n \geq 1$ and $j_0 = 2,\ j_1 = 1$. (see, [11]).

In [4], the Jacobsthal recurrence relation (1.6) is extended to higher order recurrence relations and the basic list of identities provided by A. F. Horadam [11] is expanded and extended to several identities for some of the higher order cases. In particular, third-order Jacobsthal numbers, $\{J_n^{(3)}\}_{n\geq 0}$, and third-order Jacobsthal-Lucas numbers, $\{j_n^{(3)}\}_{n\geq 0}$, are defined by

$$J_{n+3}^{(3)} = J_{n+2}^{(3)} + J_{n+1}^{(3)} + 2J_n^{(3)},\ J_0^{(3)} = 0,\ J_1^{(3)} = J_2^{(3)} = 1,\ n \geq 0, \tag{1.7}$$

and

$$j_{n+3}^{(3)} = j_{n+2}^{(3)} + j_{n+1}^{(3)} + 2j_n^{(3)},\ j_0^{(3)} = 2,\ j_1^{(3)} = 1,\ j_2^{(3)} = 5,\ n \geq 0, \tag{1.8}$$

respectively.

The following properties given for third order Jacobsthal numbers and third order Jacobsthal-Lucas numbers play important roles in this paper (for more, see [3, 4]).

$$3J_n^{(3)} + j_n^{(3)} = 2^{n+1}, \tag{1.9}$$

$$j_n^{(3)} - 3J_n^{(3)} = 2j_{n-3}^{(3)}, \tag{1.10}$$

$$J_{n+2}^{(3)} - 4J_n^{(3)} = \begin{cases} -2 & if \quad n \equiv 1 \pmod 3 \\ 1 & if \quad n \not\equiv 1 \pmod 3 \end{cases}, \tag{1.11}$$

$$j_n^{(3)} - 4J_n^{(3)} = \begin{cases} 2 & if \quad n \equiv 0 \pmod 3 \\ -3 & if \quad n \equiv 1 \pmod 3 \\ 1 & if \quad n \equiv 2 \pmod 3 \end{cases}, \tag{1.12}$$

$$j_{n+1}^{(3)} + j_n^{(3)} = 3J_{n+2}^{(3)}, \tag{1.13}$$

$$j_n^{(3)} - J_{n+2}^{(3)} = \begin{cases} 1 & if \quad n \equiv 0 \pmod 3 \\ -1 & if \quad n \equiv 1 \pmod 3 \\ 0 & if \quad n \equiv 2 \pmod 3 \end{cases}, \tag{1.14}$$

$$\left(j_{n-3}^{(3)}\right)^2 + 3J_n^{(3)} j_n^{(3)} = 4^n, \tag{1.15}$$

$$\sum_{k=0}^{n} J_k^{(3)} = \begin{cases} J_{n+1}^{(3)} & if \quad n \not\equiv 0 \pmod 3 \\ J_{n+1}^{(3)} - 1 & if \quad n \equiv 0 \pmod 3 \end{cases} \tag{1.16}$$

and

$$\left(j_n^{(3)}\right)^2 - 9\left(J_n^{(3)}\right)^2 = 2^{n+2} j_{n-3}^{(3)}. \tag{1.17}$$

Using standard techniques for solving recurrence relations, the auxiliary equation, and its roots are given by

$$x^3 - x^2 - x - 2 = 0;\ x = 2,\ and\ x = \frac{-1 \pm i\sqrt{3}}{2}.$$

Note that the latter two are the complex conjugate cube roots of unity. Call them ω_1 and ω_2, respectively. Thus the Binet formulas can be written as

$$J_n^{(3)} = \frac{1}{7}2^{n+1} - \frac{3+2i\sqrt{3}}{21}\omega_1^n - \frac{3-2i\sqrt{3}}{21}\omega_2^n = \frac{1}{7}\left(2^{n+1} - V_n^{(3)}\right) \tag{1.18}$$

and

$$j_n^{(3)} = \frac{1}{7}2^{n+3} + \frac{3+2i\sqrt{3}}{7}\omega_1^n + \frac{3-2i\sqrt{3}}{7}\omega_2^n = \frac{1}{7}\left(2^{n+3} + 3V_n^{(3)}\right), \tag{1.19}$$

respectively. Here $V_n^{(3)}$ is the sequence defined by

$$V_n^{(3)} = \frac{3+2i\sqrt{3}}{3}\omega_1^n + \frac{3-2i\sqrt{3}}{3}\omega_2^n = \begin{cases} 2 & if \quad n \equiv 0 \pmod 3 \\ -3 & if \quad n \equiv 1 \pmod 3 \\ 1 & if \quad n \equiv 2 \pmod 3 \end{cases}. \tag{1.20}$$

Using Eq. (1.20) is easy to see that for all $n \geq 0$:

$$V_n^{(3)} + 2V_{n+1}^{(3)} + 4V_{n+2}^{(3)} = \begin{cases} 0 & if \quad n \equiv 0 \pmod 3 \\ 7 & if \quad n \equiv 1 \pmod 3 \\ -7 & if \quad n \equiv 2 \pmod 3 \end{cases}.$$

Recently in [3], we have defined a new type of quaternions with the third-order Jacobsthal and third-order Jacobsthal-Lucas number components as

$$JQ_n^{(3)} = J_n^{(3)} + J_{n+1}^{(3)}\mathbf{i} + J_{n+2}^{(3)}\mathbf{j} + J_{n+3}^{(3)}\mathbf{k}$$

and

$$jQ_n^{(3)} = j_n^{(3)} + j_{n+1}^{(3)}\mathbf{i} + j_{n+2}^{(3)}\mathbf{j} + j_{n+3}^{(3)}\mathbf{k},$$

respectively, where $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$, and we studied the properties of these quaternions. Also, we derived the generating functions and many other identities for the third-order Jacobsthal and third-order Jacobsthal-Lucas quaternions.

In this paper, we define the dual third-order Jacobsthal quaternions and dual third-order Jacobsthal-Lucas quaternions as follows:

$$JN_m^{(3)} = J_m^{(3)} + J_{m+1}^{(3)}\mathbf{i} + J_{m+2}^{(3)}\mathbf{j} + J_{m+3}^{(3)}\mathbf{k} \ (m \geq 0) \tag{1.21}$$

and

$$jN_m^{(3)} = j_m^{(3)} + j_{m+1}^{(3)}\mathbf{i} + j_{m+2}^{(3)}\mathbf{j} + j_{m+3}^{(3)}\mathbf{k} \ (m \geq 0), \tag{1.22}$$

respectively. Here $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 0$, $\mathbf{ij} = -\mathbf{ji} = \mathbf{jk} = -\mathbf{kj} = \mathbf{ki} = -\mathbf{ik} = 0$. Also, we investigated the relations between the dual third-order Jacobsthal quaternions and third-order Jacobsthal numbers. Furthermore, we give some their quadratic properties, the Binet's formulas, d'Ocagne and Cassini-like identities for these quaternions.

2. Dual Third-Order Jacobsthal Quaternions

We can define dual third-order Jacobsthal quaternions by using third-order Jacobsthal numbers. The n-th third-order Jacobsthal number $J_n^{(3)}$ is defined by Eq. (1.7). Then, we can define the dual third-order Jacobsthal quaternions as follows:

$$\mathbf{N}_J = \{JN_m^{(3)} = J_m^{(3)} + J_{m+1}^{(3)}\mathbf{i} + J_{m+2}^{(3)}\mathbf{j} + J_{m+3}^{(3)}\mathbf{k} : \ m0\}, \tag{2.1}$$

where $J_m^{(3)}$ is the m-th third-order Jacobsthal number and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ as in Eq. (1.2). Also, we can define the dual third-order Jacobsthal-Lucas quaternion as follows:

$$\mathbf{N}_j = \{jN_m^{(3)} = j_m^{(3)} + j_{m+1}^{(3)}\mathbf{i} + j_{m+2}^{(3)}\mathbf{j} + j_{m+3}^{(3)}\mathbf{k} : \ m0\}, \tag{2.2}$$

where $j_m^{(3)}$ is the m-th third-order Jacobsthal-Lucas number.

Then, the addition and subtraction of the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions is defined by

$$\begin{aligned} & JN_m^{(3)} \pm jN_m^{(3)} \\ & = (J_m^{(3)} + J_{m+1}^{(3)}\mathbf{i} + J_{m+2}^{(3)}\mathbf{j} + J_{m+3}^{(3)}\mathbf{k}) \\ & \quad \pm (j_m^{(3)} + j_{m+1}^{(3)}\mathbf{i} + j_{m+2}^{(3)}\mathbf{j} + j_{m+3}^{(3)}\mathbf{k}) \\ & = (J_m^{(3)} \pm j_m^{(3)}) + (J_{m+1}^{(3)} \pm j_{m+1}^{(3)})\mathbf{i} + (J_{m+2}^{(3)} \pm j_{m+2}^{(3)})\mathbf{j} \\ & \quad + (J_{m+3}^{(3)} \pm j_{m+3}^{(3)})\mathbf{k} \end{aligned} \tag{2.3}$$

and the multiplication of the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions is defined by

$$
\begin{aligned}
&JN_m^{(3)} jN_m^{(3)} \\
&= (J_m^{(3)} + J_{m+1}^{(3)}\mathbf{i} + J_{m+2}^{(3)}\mathbf{j} + J_{m+3}^{(3)}\mathbf{k})(j_m^{(3)} + j_{m+1}^{(3)}\mathbf{i} + j_{m+2}^{(3)}\mathbf{j} + j_{m+3}^{(3)}\mathbf{k}) \\
&= J_m^{(3)} j_m^{(3)} + (J_m^{(3)} j_{m+1}^{(3)} + J_{m+1}^{(3)} j_m^{(3)})\mathbf{i} + (J_m^{(3)} j_{m+2}^{(3)} + J_{m+2}^{(3)} j_m^{(3)})\mathbf{j} \\
&\quad + (J_m^{(3)} j_{m+3}^{(3)} + J_{m+3}^{(3)} j_m^{(3)})\mathbf{k}.
\end{aligned} \tag{2.4}
$$

Now, the scalar and the vector part of the $JN_m^{(3)}$ which is the m-th term of the dual third-order Jacobsthal sequence $\{JN_m^{(3)}\}_{m\geq 0}$ are denoted by

$$
\left(S_{JN_m^{(3)}}, V_{JN_m^{(3)}}\right) = \left(J_m^{(3)}, J_{m+1}^{(3)}\mathbf{i} + J_{m+2}^{(3)}\mathbf{j} + J_{m+3}^{(3)}\mathbf{k}\right). \tag{2.5}
$$

Thus, the dual third-order Jacobsthal $JN_m^{(3)}$ is given by $S_{JN_m^{(3)}} + V_{JN_m^{(3)}}$. Then, relation (2.4) is defined by

$$
JN_m^{(3)} jN_m^{(3)} = S_{JN_m^{(3)}} S_{jN_m^{(3)}} + S_{JN_m^{(3)}} V_{jN_m^{(3)}} + S_{jN_m^{(3)}} V_{JN_m^{(3)}}. \tag{2.6}
$$

The conjugate of dual third-order Jacobsthal quaternion $JN_m^{(3)}$ is denoted by $\overline{JN}_m^{(3)}$ and it is $\overline{JN}_m^{(3)} = J_m^{(3)} - J_{m+1}^{(3)}\mathbf{i} - J_{m+2}^{(3)}\mathbf{j} - J_{m+3}^{(3)}\mathbf{k}$. The norm of $JN_m^{(3)}$ is defined as

$$
Nr^2(JN_m^{(3)}) = JN_m^{(3)} \overline{JN}_m^{(3)} = \overline{JN}_m^{(3)} JN_m^{(3)} = \left(JN_m^{(3)}\right)^2. \tag{2.7}
$$

Then, we give the following theorem using statements (2.1), (2.3) and (2.4).

Theorem 2.1. *Let $J_m^{(3)}$ and $JN_m^{(3)}$ be the m-th terms of the third-order Jacobsthal sequence $\{J_m^{(3)}\}_{m\geq 0}$ and the dual third-order Jacobsthal quaternion sequence $\{JN_m^{(3)}\}_{m\geq 0}$, respectively. In this case, for $m \geq 0$ we can give the following relations:*

$$
2JN_m^{(3)} + JN_{m+1}^{(3)} + JN_{m+2}^{(3)} = JN_{m+3}^{(3)}, \tag{2.8}
$$

$$
JN_m^{(3)} - JN_{m+1}^{(3)}\mathbf{i} - JN_{m+2}^{(3)}\mathbf{j} - JN_{m+3}^{(3)}\mathbf{k} = J_m^{(3)}, \tag{2.9}
$$

$$\left(JN_m^{(3)}\right)^2+\left(JN_{m+1}^{(3)}\right)^2+\left(JN_{m+2}^{(3)}\right)^2=\frac{1}{7}\left(\begin{array}{c}3\cdot 2^{2(m+1)}(1+4\mathbf{i}+8\mathbf{j}+16\mathbf{k})\\ -2^{m+2}UN_m^{(3)}\\ -2^{m+3}U_m^{(3)}(\mathbf{i}+2\mathbf{j}+4\mathbf{k})\\ +2(1-\mathbf{i}-\mathbf{j}+2\mathbf{k})\end{array}\right),$$

(2.10)

where

$$UN_m^{(3)}=U_m^{(3)}+U_{m+1}^{(3)}\mathbf{i}+U_{m+2}^{(3)}\mathbf{j}+U_{m+3}^{(3)}\mathbf{k}\ \textit{and}\ U_m^{(3)}=\tfrac{1}{7}\left(V_{m+1}^{(3)}+3V_{m+2}^{(3)}\right).$$

Proof. (2.8): By the equations $JN_m^{(3)}=J_m^{(3)}+J_{m+1}^{(3)}\mathbf{i}+J_{m+2}^{(3)}\mathbf{j}+J_{m+3}^{(3)}\mathbf{k}$ and (1.7), we get $2\mathrm{JN}_m^{(3)}+JN_{m+1}^{(3)}+JN_{m+2}^{(3)}$

$= (2J_m^{(3)}+2J_{m+1}^{(3)}\mathbf{i}+2J_{m+2}^{(3)}\mathbf{j}+2J_{m+3}^{(3)}\mathbf{k})$
$+ (J_{m+1}^{(3)}+J_{m+2}^{(3)}\mathbf{i}+J_{m+3}^{(3)}\mathbf{j}+J_{m+4}^{(3)}\mathbf{k})$
$+ (J_{m+2}^{(3)}+J_{m+3}^{(3)}\mathbf{i}+J_{m+4}^{(3)}\mathbf{j}+J_{m+5}^{(3)}\mathbf{k})$
$= (2J_m^{(3)}+J_{m+1}^{(3)}+J_{m+2}^{(3)})+(2J_{m+1}^{(3)}+J_{m+2}^{(3)}+J_{m+3}^{(3)})\mathbf{i}$
$+ (2J_{m+2}^{(3)}+J_{m+3}^{(3)}+J_{m+4}^{(3)})\mathbf{j}+(2J_{m+3}^{(3)}+J_{m+4}^{(3)}+J_{m+5}^{(3)})\mathbf{k}$
$= J_{m+3}^{(3)}+J_{m+4}^{(3)}\mathbf{i}+J_{m+5}^{(3)}\mathbf{j}+J_{m+6}^{(3)}\mathbf{k}$
$= JN_{m+3}^{(3)}.$

(2.9): By using $JN_m^{(3)}$ in the Eq. (2.1) and conditions (1.2), we get

$\mathrm{JN}_m^{(3)}-JN_{m+1}^{(3)}\mathbf{i}-JN_{m+2}^{(3)}\mathbf{j}-JN_{m+3}^{(3)}\mathbf{k}$
$= J_m^{(3)}+J_{m+1}^{(3)}\mathbf{i}+J_{m+2}^{(3)}\mathbf{j}+J_{m+3}^{(3)}\mathbf{k}$
$- (J_{m+1}^{(3)}+J_{m+2}^{(3)}\mathbf{i}+J_{m+3}^{(3)}\mathbf{j}+J_{m+4}^{(3)}\mathbf{k})\mathbf{i}$
$- (J_{m+2}^{(3)}+J_{m+3}^{(3)}\mathbf{i}+J_{m+4}^{(3)}\mathbf{j}+J_{m+5}^{(3)}\mathbf{k})\mathbf{j}$
$- (J_{m+3}^{(3)}+J_{m+4}^{(3)}\mathbf{i}+J_{m+5}^{(3)}\mathbf{j}+J_{m+6}^{(3)}\mathbf{k})\mathbf{k}$
$= J_m^{(3)}.$

(2.10): By using Eqs. (2.4) and (1.18), we get

(2.11) $$\left(JN_m^{(3)}\right)^2=\left(J_m^{(3)}\right)^2+2J_m^{(3)}J_{m+1}^{(3)}\mathbf{i}+2J_m^{(3)}J_{m+2}^{(3)}\mathbf{j}+2J_m^{(3)}J_{m+3}^{(3)}\mathbf{k}$$

and

$$\begin{aligned}
&\left(J_m^{(3)}\right)^2 \\
&+\left(J_{m+1}^{(3)}\right)^2+\left(J_{m+2}^{(3)}\right)^2 \\
&= \tfrac{1}{49}\left(\left(2^{m+1}-V_m^{(3)}\right)^2+\left(2^{m+2}-V_{m+1}^{(3)}\right)^2+\left(2^{m+3}-V_{m+2}^{(3)}\right)^2\right) \quad (2.12)\\
&= \tfrac{1}{49}\left(21\cdot 2^{2(m+1)}-2^{m+2}(V_m^{(3)}+2V_{m+1}^{(3)}+4V_{m+2}^{(3)})+14\right)\\
&= \tfrac{1}{7}\left(3\cdot 2^{2(m+1)}-2^{m+2}U_m^{(3)}+2\right),
\end{aligned}$$

where $U_m^{(3)} = \frac{1}{7}\left(V_m^{(3)}+2V_{m+1}^{(3)}+4V_{m+2}^{(3)}\right) = \frac{1}{7}\left(V_{m+1}^{(3)}+3V_{m+2}^{(3)}\right)$. Finally, from the Eqs. (2.11) and (2.12), we obtain

$$\begin{aligned}
&\left(JN_m^{(3)}\right)^2+\left(JN_{m+1}^{(3)}\right)^2+\left(JN_{m+2}^{(3)}\right)^2\\
&=\left(J_m^{(3)}\right)^2+\left(J_{m+1}^{(3)}\right)^2+\left(J_{m+2}^{(3)}\right)^2\\
&+2\left(J_m^{(3)}J_{m+1}^{(3)}+J_{m+1}^{(3)}J_{m+2}^{(3)}+J_{m+2}^{(3)}J_{m+3}^{(3)}\right)\mathbf{i}\\
&+2\left(J_m^{(3)}J_{m+2}^{(3)}+J_{m+1}^{(3)}J_{m+3}^{(3)}+J_{m+2}^{(3)}J_{m+4}^{(3)}\right)\mathbf{j}\\
&+2\left(J_m^{(3)}J_{m+3}^{(3)}+J_{m+1}^{(3)}J_{m+4}^{(3)}+J_{m+2}^{(3)}J_{m+5}^{(3)}\right)\mathbf{k}\\
&=\tfrac{1}{7}\left(\begin{array}{c} 3\cdot 2^{2(m+1)}(1+4\mathbf{i}+8\mathbf{j}+16\mathbf{k})-2^{m+2}UN_m^{(3)}\\ -2^{m+3}U_m^{(3)}(\mathbf{i}+2\mathbf{j}+4\mathbf{k})+2(1-\mathbf{i}-\mathbf{j}+2\mathbf{k})\end{array}\right),
\end{aligned}$$

where $UN_m^{(3)} = U_m^{(3)} + U_{m+1}^{(3)}\mathbf{i} + U_{m+2}^{(3)}\mathbf{j} + U_{m+3}^{(3)}\mathbf{k}$. □

Theorem 2.2. *Let $JN_m^{(3)}$ and $jN_m^{(3)}$ be the m-th terms of the dual third-order Jacobsthal quaternion sequence $\{JN_m^{(3)}\}_{m\geq 0}$ and the dual third-order Jacobsthal-Lucas quaternion sequence $\{jN_m^{(3)}\}_{m\geq 0}$, respectively. The following relations are satisfied*

$$jN_{m+3}^{(3)} - 3JN_{m+3}^{(3)} = 2jN_m^{(3)}, \quad (2.13)$$

$$jN_{m+1}^{(3)} + jN_m^{(3)} = 3JN_{m+2}^{(3)}, \quad (2.14)$$

$$\left(jN_m^{(3)}\right)^2 + 3JN_{m+3}^{(3)}jN_{m+3}^{(3)} = 4^{m+3}(1+4\mathbf{i}+8\mathbf{j}+16\mathbf{k}). \quad (2.15)$$

Proof. (2.13): From identities between third-order Jacobsthal number and third-order Jacobsthal-Lucas number (1.10) and (2.3), it follows that

$$\begin{aligned}
&\mathrm{jN}^{(3)}_{m+3} - 3JN^{(3)}_{m+3} = j^{(3)}_{m+3} + j^{(3)}_{m+4}\mathbf{i} + j^{(3)}_{m+5}\mathbf{j} + j^{(3)}_{m+6}\mathbf{k}\\
&- 3(J^{(3)}_{m+3} + J^{(3)}_{m+4}\mathbf{i} + J^{(3)}_{m+5}\mathbf{j} + J^{(3)}_{m+6}\mathbf{k})\\
&= (j^{(3)}_{m+3} - 3J^{(3)}_{m+3}) + (j^{(3)}_{m+4} - 3J^{(3)}_{m+4})\mathbf{i}\\
&+ (j^{(3)}_{m+5} - 3J^{(3)}_{m+5})\mathbf{j} + (j^{(3)}_{m+6} - 3J^{(3)}_{m+6})\mathbf{k}\\
&= 2j^{(3)}_{m} + 2j^{(3)}_{m+1}\mathbf{i} + 2j^{(3)}_{m+2}\mathbf{j} + 2j^{(3)}_{m+3}\mathbf{k}\\
&= 2jN^{(3)}_{m}.
\end{aligned}$$

The proof of (2.14) is similar to (2.13), using the identity (1.13). (2.15): Now, using Eqs. (2.4), (2.11) and (1.15), we get $\left(jN^{(3)}_{m}\right)^2 + 3JN^{(3)}_{m+3}jN^{(3)}_{m+3}$

$$\begin{aligned}
&= \left(j^{(3)}_{m}\right)^2 + 2j^{(3)}_{m}j^{(3)}_{m+1}\mathbf{i} + 2j^{(3)}_{m}j^{(3)}_{m+2}\mathbf{j} + 2j^{(3)}_{m}j^{(3)}_{m+3}\mathbf{k}\\
&+ 3J^{(3)}_{m+3}j^{(3)}_{m+3} + 3(J^{(3)}_{m+3}j^{(3)}_{m+4} + J^{(3)}_{m+4}j^{(3)}_{m+3})\mathbf{i}\\
&+ 3(J^{(3)}_{m+3}j^{(3)}_{m+5} + J^{(3)}_{m+5}j^{(3)}_{m+3})\mathbf{j} + 3(J^{(3)}_{m+3}j^{(3)}_{m+6} + J^{(3)}_{m+6}j^{(3)}_{m+3})\mathbf{k}\\
&= \left(j^{(3)}_{m}\right)^2 + 3J^{(3)}_{m+3}j^{(3)}_{m+3} + \left(2j^{(3)}_{m}j^{(3)}_{m+1} + 3(J^{(3)}_{m+3}j^{(3)}_{m+4} + J^{(3)}_{m+4}j^{(3)}_{m+3})\right)\mathbf{i}\\
&+ \left(2j^{(3)}_{m}j^{(3)}_{m+2} + 3(J^{(3)}_{m+3}j^{(3)}_{m+5} + J^{(3)}_{m+5}j^{(3)}_{m+3})\right)\mathbf{j}\\
&+ \left(2j^{(3)}_{m}j^{(3)}_{m+3} + 3(J^{(3)}_{m+3}j^{(3)}_{m+6} + J^{(3)}_{m+6}j^{(3)}_{m+3})\right)\mathbf{k}\\
&= 4^{m+3}(1 + 4\mathbf{i} + 8\mathbf{j} + 16\mathbf{k}),
\end{aligned}$$

the last equality because $3J^{(3)}_{m+3}j^{(3)}_{m+3} = 4^{m+3} - \left(j^{(3)}_{m}\right)^2$ in Eq. (1.15). □

Theorem 2.3. *Let $JN^{(3)}_{m}$ be the m-th term of the dual third-order Jacobsthal quaternion sequence $\{JN^{(3)}_{m}\}_{m\geq 0}$. Then, we have the following identity*

$$\sum_{s=0}^{m} JN^{(3)}_{s} = JN^{(3)}_{m+1} - \frac{1}{21}\left(7(1 + \mathbf{i} + 4\mathbf{j} + 7\mathbf{k}) - 4VN^{(3)}_{m+1} + VN^{(3)}_{m}\right), \tag{2.16}$$

where $VN^{(3)}_{m} = V^{(3)}_{m} + V^{(3)}_{m+1}\mathbf{i} + V^{(3)}_{m+2}\mathbf{j} + V^{(3)}_{m+3}\mathbf{k}$.

Proof. Since

$$\sum_{s=0}^{m} J^{(3)}_{s} = J^{(3)}_{m+1} - \frac{1}{21}\left(7 - 4V^{(3)}_{m+1} + V^{(3)}_{m}\right) = \begin{cases} J^{(3)}_{m+1} & if \quad n \not\equiv 0 \pmod 3\\ J^{(3)}_{m+1} - 1 & if \quad n \equiv 0 \pmod 3\end{cases} \tag{2.17}$$

(see [3]), we get

$\sum_{s=0}^{m} JN_s^{(3)} = \sum_{s=0}^{m} J_s^{(3)} + \mathbf{i}\sum_{s=1}^{m+1} J_s^{(3)} + \mathbf{j}\sum_{s=2}^{m+2} J_s^{(3)} + \mathbf{k}\sum_{s=3}^{m+3} J_s^{(3)}$
$= J_{m+1}^{(3)} - \frac{1}{21}\left(7 - 4V_{m+1}^{(3)} + V_m^{(3)}\right)$
$+ \left(J_{m+2}^{(3)} - \frac{1}{21}\left(7 - 4V_{m+2}^{(3)} + V_{m+1}^{(3)}\right)\right)\mathbf{i}$
$+ \left(J_{m+3}^{(3)} - \frac{1}{21}\left(28 - 4V_{m+3}^{(3)} + V_{m+2}^{(3)}\right)\right)\mathbf{j}$
$+ \left(J_{m+4}^{(3)} - \frac{1}{21}\left(49 - 4V_{m+4}^{(3)} + V_{m+3}^{(3)}\right)\right)\mathbf{k}$
$= JN_{m+1}^{(3)} - \frac{1}{21}\left(7(1 + \mathbf{i} + 4\mathbf{j} + 7\mathbf{k}) - 4VN_{m+1}^{(3)} + VN_m^{(3)}\right),$
where $VN_m^{(3)} = V_m^{(3)} + V_{m+1}^{(3)}\mathbf{i} + V_{m+2}^{(3)}\mathbf{j} + V_{m+3}^{(3)}\mathbf{k}$. □

Theorem 2.4. *Let $JN_m^{(3)}$ and $jN_m^{(3)}$ be the m-th terms of the dual third-order Jacobsthal quaternion sequence $\{JN_m^{(3)}\}_{m\geq 0}$ and the dual third-order Jacobsthal-Lucas quaternion sequence $\{jN_m^{(3)}\}_{m\geq 0}$, respectively. Then, we have*

$$jN_m^{(3)}\overline{JN}_m^{(3)} - \overline{jN}_m^{(3)} JN_m^{(3)} = 2(J_m^{(3)} jN_m^{(3)} - j_m^{(3)} JN_m^{(3)}), \tag{2.18}$$

$$jN_m^{(3)} JN_m^{(3)} + \overline{jN}_m^{(3)}\overline{JN}_m^{(3)} = 2j_m^{(3)} J_m^{(3)}. \tag{2.19}$$

Proof. (2.18): By the Eqs. (2.1), (2.2) and $\overline{JN}_m^{(3)} = J_m^{(3)} - J_{m+1}^{(3)}\mathbf{i} - J_{m+2}^{(3)}\mathbf{j} - J_{m+3}^{(3)}\mathbf{k}$, we get $\mathrm{jN}_m^{(3)}\overline{JN}_m^{(3)} - \overline{jN}_m^{(3)} JN_m^{(3)}$
$= (j_m^{(3)} + j_{m+1}^{(3)}\mathbf{i} + j_{m+2}^{(3)}\mathbf{j} + j_{m+3}^{(3)}\mathbf{k})(J_m^{(3)} - J_{m+1}^{(3)}\mathbf{i} - J_{m+2}^{(3)}\mathbf{j} - J_{m+3}^{(3)}\mathbf{k})$
$- (j_m^{(3)} - j_{m+1}^{(3)}\mathbf{i} - j_{m+2}^{(3)}\mathbf{j} - j_{m+3}^{(3)}\mathbf{k})(J_m^{(3)} + J_{m+1}^{(3)}\mathbf{i} + J_{m+2}^{(3)}\mathbf{j} + J_{m+3}^{(3)}\mathbf{k})$
$= 2J_m^{(3)}(j_{m+1}^{(3)}\mathbf{i} + j_{m+2}^{(3)}\mathbf{j} + j_{m+3}^{(3)}\mathbf{k}) - 2j_m^{(3)}(J_{m+1}^{(3)}\mathbf{i} + J_{m+2}^{(3)}\mathbf{j} + J_{m+3}^{(3)}\mathbf{k})$
$= 2(J_m^{(3)} jN_m^{(3)} - j_m^{(3)} JN_m^{(3)})$.

(2.19): $\mathrm{jN}_m^{(3)} JN_m^{(3)} + \overline{jN}_m^{(3)}\overline{JN}_m^{(3)}$
$= (j_m^{(3)} + j_{m+1}^{(3)}\mathbf{i} + j_{m+2}^{(3)}\mathbf{j} + j_{m+3}^{(3)}\mathbf{k})(J_m^{(3)} + J_{m+1}^{(3)}\mathbf{i} + J_{m+2}^{(3)}\mathbf{j} + J_{m+3}^{(3)}\mathbf{k})$
$+ (j_m^{(3)} - j_{m+1}^{(3)}\mathbf{i} - j_{m+2}^{(3)}\mathbf{j} - j_{m+3}^{(3)}\mathbf{k})(J_m^{(3)} - J_{m+1}^{(3)}\mathbf{i} - J_{m+2}^{(3)}\mathbf{j} - J_{m+3}^{(3)}\mathbf{k})$
$= j_m^{(3)} J_m^{(3)} + (j_m^{(3)} J_{m+1}^{(3)} + j_{m+1}^{(3)} J_m^{(3)})\mathbf{i} + (j_m^{(3)} J_{m+2}^{(3)} + j_{m+2}^{(3)} J_m^{(3)})\mathbf{j}$
$+ (j_m^{(3)} J_{m+3}^{(3)} + j_{m+3}^{(3)} J_m^{(3)})\mathbf{k}$
$+ j_m^{(3)} J_m^{(3)} - (j_m^{(3)} J_{m+1}^{(3)} + j_{m+1}^{(3)} J_m^{(3)})\mathbf{i} - (j_m^{(3)} J_{m+2}^{(3)} + j_{m+2}^{(3)} J_m^{(3)})\mathbf{j}$
$- (j_m^{(3)} J_{m+3}^{(3)} + j_{m+3}^{(3)} J_m^{(3)})\mathbf{k}$
$= 2j_m^{(3)} J_m^{(3)}$. □

Theorem 2.5 (Binet's Formulas). *Let $JN_m^{(3)}$ and $jN_m^{(3)}$ be m-th terms of the dual third-order Jacobsthal quaternion sequence $\{JN_m^{(3)}\}_{m\geq 0}$ and the dual third-order Jacobsthal-Lucas quaternion sequence $\{JN_m^{(3)}\}_{m\geq 0}$, respectively. For $m \geq 0$, the Binet's formulas for these quaternions are as follows:*

$$JN_m^{(3)} = \frac{1}{7}2^{m+1}\underline{\alpha} - \frac{3+2i\sqrt{3}}{21}\underline{\omega_1}\omega_1^m - \frac{3-2i\sqrt{3}}{21}\underline{\omega_2}\omega_2^m = \frac{1}{7}\left(2^{m+1}\underline{\alpha} - VN_m^{(3)}\right) \tag{2.20}$$

and

$$jN_m^{(3)} = \frac{1}{7}2^{m+3}\underline{\alpha} + \frac{3+2i\sqrt{3}}{7}\underline{\omega_1}\omega_1^m + \frac{3-2i\sqrt{3}}{7}\underline{\omega_2}\omega_2^m = \frac{1}{7}\left(2^{m+3}\underline{\alpha} + 3VN_m^{(3)}\right), \tag{2.21}$$

respectively, where $VN_m^{(3)}$ is the sequence defined by

$$VN_m^{(3)} = \begin{cases} 2-3\mathbf{i}+\mathbf{j}+2\mathbf{k} & if \quad n \equiv 0 \pmod 3 \\ -3+\mathbf{i}+2\mathbf{j}-3\mathbf{k} & if \quad n \equiv 1 \pmod 3 \\ 1+2\mathbf{i}-3\mathbf{j}+\mathbf{k} & if \quad n \equiv 2 \pmod 3 \end{cases}, \tag{2.22}$$

$\underline{\alpha} = 1+2\mathbf{i}+4\mathbf{j}+8\mathbf{k}$ *and* $\underline{\omega_{1,2}} = 1+\omega_{1,2}\mathbf{i}+\omega_{1,2}^2\mathbf{j}+\mathbf{k}$.

Proof. Repeated use of (1.18) in (2.1) enables one to write for $\underline{\alpha} = 1+2\mathbf{i}+4\mathbf{j}+8\mathbf{k}$ and $\underline{\omega_{1,2}} = 1+\omega_{1,2}\mathbf{i}+\omega_{1,2}^2\mathbf{j}+\mathbf{k}$,

$$\begin{aligned} & JN_m^{(3)} \\ &= J_m^{(3)} + J_{m+1}^{(3)}\mathbf{i} + J_{m+2}^{(3)}\mathbf{j} + J_{m+3}^{(3)}\mathbf{k} \\ &= \tfrac{1}{7}2^{m+1} - \tfrac{3+2i\sqrt{3}}{21}\omega_1^m - \tfrac{3-2i\sqrt{3}}{21}\omega_2^m \\ &\quad + \left(\tfrac{1}{7}2^{m+2} - \tfrac{3+2i\sqrt{3}}{21}\omega_1^{m+1} - \tfrac{3-2i\sqrt{3}}{21}\omega_2^{m+1}\right)\mathbf{i} \\ &\quad + \left(\tfrac{1}{7}2^{m+3} - \tfrac{3+2i\sqrt{3}}{21}\omega_1^{m+2} - \tfrac{3-2i\sqrt{3}}{21}\omega_2^{m+2}\right)\mathbf{j} \\ &\quad + \left(\tfrac{1}{7}2^{m+4} - \tfrac{3+2i\sqrt{3}}{21}\omega_1^{m+3} - \tfrac{3-2i\sqrt{3}}{21}\omega_2^{m+3}\right)\mathbf{k} \\ &= \tfrac{1}{7}2^{m+1}\underline{\alpha} + \tfrac{3+2i\sqrt{3}}{7}\underline{\omega_1}\omega_1^m + \tfrac{3-2i\sqrt{3}}{7}\underline{\omega_2}\omega_2^m \end{aligned} \tag{2.23}$$

and similarly making use of (1.19) in (2.2) yields

$$jN_m^{(3)} = j_m^{(3)} + j_{m+1}^{(3)}\mathbf{i} + j_{m+2}^{(3)}\mathbf{j} + j_{m+3}^{(3)}\mathbf{k} = \frac{1}{7}2^{m+3}\underline{\alpha} + \frac{3+2i\sqrt{3}}{7}\underline{\omega_1}\omega_1^m + \frac{3-2i\sqrt{3}}{7}\underline{\omega_2}\omega_2^m. \tag{2.24}$$

The formulas in (2.23) and (2.24) are called as Binet's formulas for the dual third-order Jacobsthal and dual third-order Jacobsthal-Lucas quaternions, respectively. Using notation in (2.22), we obtain the results (2.20) and (2.21). □

Theorem 2.6 (D'Ocagne-like Identity). *Let $JN_m^{(3)}$ be the m-th terms of the dual third-order Jacobsthal quaternion sequence $\{JN_m^{(3)}\}_{m\geq 0}$. In this case, for $n \geq m \geq 0$, the d'Ocagne identities for $JN_m^{(3)}$ is as follows:*

$$JN_n^{(3)}JN_{m+1}^{(3)} - JN_{n+1}^{(3)}JN_m^{(3)} = \frac{1}{7}\left(\begin{array}{c} \underline{\alpha}\left(2^{n+1}UN_{m+1}^{(3)} - 2^{m+1}UN_{n+1}^{(3)}\right) \\ +(1-\mathbf{i}-\mathbf{j}+2\mathbf{k})U_{n-m}^{(3)} \end{array}\right), \tag{2.25}$$

$$\left(JN_{m+1}^{(3)}\right)^2 - JN_{m+2}^{(3)}JN_m^{(3)} = \frac{1}{7}\left(\begin{array}{c} 2^{m+1}\underline{\alpha}(2UN_{m+1}^{(3)} - UN_{m+2}^{(3)}) \\ +(1-\mathbf{i}-\mathbf{j}+2\mathbf{k}) \end{array}\right), \tag{2.26}$$

where $UN_{m+1}^{(3)} = \frac{1}{7}(2VN_m^{(3)} - VN_{m+1}^{(3)})$, $\underline{\alpha} = 1 + 2\mathbf{i} + 4\mathbf{j} + 8\mathbf{k}$ and $U_n^{(3)}$ as in Eq. (2.12).

Proof. (2.25): Using Eqs. (2.20) and (2.22), we get

$$\begin{aligned}
&JN_n^{(3)}JN_{m+1}^{(3)}-JN_{n+1}^{(3)}JN_m^{(3)}\\
&=\tfrac{1}{49}\begin{pmatrix} \left(2^{n+1}\underline{\alpha}-VN_n^{(3)}\right)\left(2^{m+2}\underline{\alpha}-VN_{m+1}^{(3)}\right)\\ -\left(2^{n+2}\underline{\alpha}-VN_{n+1}^{(3)}\right)\left(2^{m+1}\underline{\alpha}-VN_m^{(3)}\right)\end{pmatrix}\\
&=\tfrac{1}{49}\begin{pmatrix} 2^{n+m+3}\underline{\alpha}^2-2^{n+1}\underline{\alpha}VN_{m+1}^{(3)}-2^{m+2}VN_n^{(3)}\underline{\alpha}+VN_n^{(3)}VN_{m+1}^{(3)}\\ -2^{n+m+3}\underline{\alpha}^2+2^{n+2}\underline{\alpha}VN_m^{(3)}+2^{m+1}VN_{n+1}^{(3)}\underline{\alpha}-VN_{n+1}^{(3)}VN_m^{(3)}\end{pmatrix}\\
&=\tfrac{1}{7}\left(\ \underline{\alpha}\left(2^{n+1}UN_{m+1}^{(3)}-2^{m+1}UN_{n+1}^{(3)}\right)+(1-\mathbf{i}-\mathbf{j}+2\mathbf{k})U_{n-m}^{(3)}\ \right),
\end{aligned}\tag{2.27}$$

where $UN_{m+1}^{(3)}=\frac{1}{7}(2VN_m^{(3)}-VN_{m+1}^{(3)})$ and $VN_m^{(3)}$ as in (2.22). In particular, if $n=m+1$ in Eq. (2.27), we obtain for $m\geq 0$,

$$\left(JN_{m+1}^{(3)}\right)^2-JN_{m+2}^{(3)}JN_m^{(3)}=\frac{1}{7}\begin{pmatrix} 2^{m+1}\underline{\alpha}(2UN_{m+1}^{(3)}-UN_{m+2}^{(3)})\\ +(1-\mathbf{i}-\mathbf{j}+2\mathbf{k})\end{pmatrix}.\tag{2.28}$$

□

We will give an example in which we check in a particular case the Cassini-like identity for dual third-order Jacobsthal quaternions.

Example 2.7. *Let* $\{JN_s^{(3)}:\ s=0,1,2,3\}$ *be the dual third-order Jacobsthal quaternions such that* $JN_0^{(3)}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$, $JN_1^{(3)}=1+\mathbf{i}+2\mathbf{j}+5\mathbf{k}$, $JN_2^{(3)}=1+2\mathbf{i}+5\mathbf{j}+9\mathbf{k}$ *and* $JN_3^{(3)}=2+5\mathbf{i}+9\mathbf{j}+18\mathbf{k}$. *In this case,*

$$\begin{aligned}
&\left(JN_1^{(3)}\right)^2-JN_2^{(3)}JN_0^{(3)}\\
&=(1+\mathbf{i}+2\mathbf{j}+5\mathbf{k})^2-(1+2\mathbf{i}+5\mathbf{j}+9\mathbf{k})(\mathbf{i}+\mathbf{j}+2\mathbf{k})\\
&=(1+2\mathbf{i}+4\mathbf{j}+10\mathbf{k})-(\mathbf{i}+\mathbf{j}+2\mathbf{k})\\
&=1+\mathbf{i}+3\mathbf{j}+8\mathbf{k}\\
&=\tfrac{1}{7}\begin{pmatrix} 2(1+2\mathbf{i}+4\mathbf{j}+8\mathbf{k})(2UN_1^{(3)}-UN_2^{(3)})\\ +(1-\mathbf{i}-\mathbf{j}+2\mathbf{k})\end{pmatrix}
\end{aligned}$$

and

$$
\begin{aligned}
&\left(JN_2^{(3)}\right)^2 - JN_3^{(3)}JN_1^{(3)} \\
&= (1+2\mathbf{i}+5\mathbf{j}+9\mathbf{k})^2 - (2+5\mathbf{i}+9\mathbf{j}+18\mathbf{k})(1+\mathbf{i}+2\mathbf{j}+5\mathbf{k}) \\
&= (1+4\mathbf{i}+10\mathbf{j}+18\mathbf{k}) - (2+7\mathbf{i}+13\mathbf{j}+28\mathbf{k}) \\
&= -1-3\mathbf{i}-3\mathbf{j}-10\mathbf{k} \\
&= \tfrac{1}{7}\left(\begin{array}{c} 4(1+2\mathbf{i}+4\mathbf{j}+8\mathbf{k})(2UN_2^{(3)} - UN_3^{(3)}) \\ +(1-\mathbf{i}-\mathbf{j}+2\mathbf{k}) \end{array}\right).
\end{aligned}
$$

3. Conclusions

There are two differences between the dual third-order Jacobsthal and the dual coefficient third-order Jacobsthal quaternions. The first one is as follows: the dual coefficient third-order Jacobsthal quaternionic units are $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$ whereas the dual third-order Jacobsthal quaternionic units are $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = 0,\ \mathbf{ij} = -\mathbf{ji} = \mathbf{jk} = -\mathbf{kj} = \mathbf{ki} = -\mathbf{ik} = 0$. The second one is as follows: the elements of the dual coefficient third-order Jacobsthal quaternion are $J_m^{(3)} + \varepsilon J_{m+1}^{(3)}$ ($\varepsilon^2 = 0,\ \varepsilon \neq 0$) whereas the elements of the dual third-order Jacobsthal quaternions are m-th third-order Jacobsthal number $J_m^{(3)}$.

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