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Quiver representations and their applications

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Abstract

In this article, we survey some results on geometric methods to study quiver representations, and applications of these results to sheaves, equivariant sheaves and parabolic bundles.

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Introduction

The theory of quivers and their representations has been an active area of research for many years and found applications in many other branches of Mathematics such as algebra, Lie theory, algebraic geometry, and even in Physics. The main problem lies in the classification of the representations of a given quiver, up to isomorphism. A theorem due to Gabriel completely solved the problem in a very special case, for quivers of simply-laced Dynkin type, they admit only finitely many isomorphism classes of indecomposable representations. In general, it is very difficult to find a classification result for arbitrary quivers.

Geometrically, the problem translates into the study of the orbit space of a certain affine space under the action of a reductive group. However as we shall see, simply considering the quotient obtained by classical invariant theory does not always lead to an interesting moduli space: the quotient is trivial for quivers without oriented cycles.

In order to get an interesting moduli space of representations of quivers, A. King (1994) [18] introduced the notion of stability for representations of a quiver in the same vein as the notion of stability for vector bundles due to D. Mumford. The idea is to pick an open subset that contains enough closed orbits, and to construct the quotient of that subset via Mumford's Geometric Invariant Theory (GIT). The choice of the open set depends on a notion of stability, which has both algebraic and geometric interpretations.

The moduli spaces for vector bundles or locally free sheaves on smooth projective curves were first constructed by Mumford (1963) [28] and Seshadri (1967) [36] by introducing several key ideas and the notions like, stability, semistability and S-equivalence. These notions and constructions were extended to higher dimensional projective varieties over arbitrary algebraically closed fields [17, 21, 22]. The work of Simpson [39] and Langer [20] accomplished the construction of moduli of sheaves over higher dimensional varieties, using methods of Mumford's Geometric Invariant Theory. For a modern account of moduli spaces of sheaves and their construction in higher dimensions, see [19].

In [1], Luis Álvarez-Cónsul and Alastair King have constructed the moduli spaces of semistable sheaves using the representations of a Kronecker quiver. This new approach provides an explicit closed scheme-theoretic embedding of moduli spaces $\mathcal{M}_X^{ss}(P)$ of semistable sheaves having fixed Hilbert polynomial P using certain determinant theta functions on such moduli spaces. These determinant theta functions coincide with the Faltings theta functions on moduli of semistable vector bundles on smooth projective curves. This positively answers the question of C. S. Seshadri related to Faltings theta functions (see Section 4.1).

In [2], the authors have extended a functorial construction of [1] to the moduli of equivariant sheaves on projective Γ-schemes, for a finite group Γ, by introducing the Kronecker-McKay quiver.

In [5], a GIT-free construction of the moduli space of semistable parabolic bundles over a smooth projective curve is constructed using the analogous Faltings parabolic theta functions. In [3], it is proved that Faltings parabolic theta functions can be used to give a closed scheme-theoretic embedding of moduli of semistable parabolic bundles, using the results of [2].

In this article, our main aim is to survey the results of the papers [13, 40, 18, 1, 38, 5, 2, 3] by outlining the ideas rather than reproducing the formal proofs. The literature on the topics discussed here is enormous and the list of references given at the end of this article is by no means complete, the author wishes to apologize for the unintentional exclusions of missing relevant references.

The article is organized as follows: In Section 1, we recall some basic definitions and results pertaining to quivers and their representations. In Section 2, we present the main results of [13, 40] concerning the ring of semi-invariants of quivers. In Section 3, we briefly outline the results of [18] regarding the construction of quiver representation using Geometric Invariant Theory. Section 4 is devoted to some applications of quiver representations to moduli of sheaves and closely related theta functions.

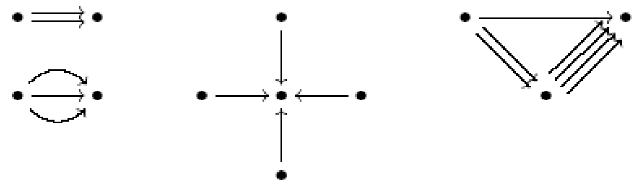
1. Preliminaries

In this section, we will recall some basic concepts related to quivers and their representations. The material of this section is mainly taken from [4, 6, 9, 11].

Quiver representations

A quiver is just a (finite) directed graph. More precisely, a quiver is a quadruple $Q = (Q_0, Q_1, s, t)$, consisting of the finite set of vertices Q_0, the finite set of arrows Q_1, the source and the target maps $s, t : Q_1 \to Q_0$.

Example 1.1.



Definition 1.2. A representation M of a quiver Q consists of a family of k-vector spaces M_v indexed by the vertices $v \in Q_0$ together with a family of linear maps $M_a\colon M_{s(a)} \longrightarrow M_{t(a)}$ indexed by the arrows $a \in Q_1$.

A representation M of a quiver Q is finite dimensional if so are all the vector spaces M_v. In this case, the family

$$\underline{\dim}(M) := (\dim M_v)_{v \in Q_0} \in \mathbf{Z}_{\geq 0}^{Q_0}$$

is the dimension vector of M.

Let M and N be two representations of a quiver Q. A morphism $\phi\colon M \longrightarrow N$ is a family of k-linear maps

$$(\phi_v\colon M_v \longrightarrow N_v)_{v \in Q_0}$$

such that the diagram

$$\begin{array}{ccc} M_{s(a)} & \xrightarrow{M_a} & M_{t(a)} \\ \phi_{s(a)} \downarrow & & \downarrow \phi_{t(a)} \\ N_{s(a)} & \xrightarrow[N_a]{} & N_{t(a)} \end{array}$$

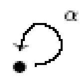
commutes for each $a \in Q_1$.

We say that a morphism $\phi\colon M \longrightarrow N$ is an isomorphism if for each $v \in Q_0$, the linear map $\phi_v\colon M_v \longrightarrow N_v$ is an isomorphism of k-vector spaces, and we write $M \cong N$.

We denote by $Hom_Q(M, N)$ the set of morphisms of representations from M to N. In fact, we get the category of representations of a quiver Q

over k, which we denote by $\mathbf{Rep}_k(Q)$. One can define the usual operations like, direct sums, subrepresentations, quotients of representations in an obvious way. It is easy to check that $\mathbf{Rep}_k(Q)$ is k-linear abelian category.

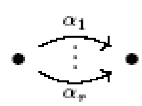
Example 1.3. *Consider the loop quiver*

$L:$ 

A representation of L is a pair (V, f); where V a vector space of dimension n and f an endomorphism of V. By choosing a basis of V, we can identify f with $n \times n$ matrix A. Therefore, the isomorphism classes of n-dimensional representations of L correspond bijectively to the conjugacy classes of $n \times n$ matrices. In particular, there are infinitely many isomorphism classes of representations of the loop having a prescribed dimension.

More generally, if we consider the r-loop L_r (a quiver with one vertex and r loops), then there are infinitely many isomorphism classes of representations of L_r having a prescribed dimension.

Example 1.4. Consider the r-arrow Kronecker quiver

$K_r:$ 

A representation of K_r consists two vector spaces V, W together with r linear maps

$$f_1, \dots, f_r : V \longrightarrow W .$$

It is clear that when $r = 1$, then the the representations of dimension vector (m, n) are classified by the rank of the $n \times m$ matrix; and hence there are only finitely many isomorphism classes of representations of K_1 having dimension vector (m, n). While in the case of $r \geq 2$, the classification of representations of K_r upto isomorphism is quite complicated. In this case, there are infinitely many isomorphism classes of representations of K_r with prescribed dimension vector. It is worth to note that when $r = 2$, then the classification is due to Kronecker (see [6, Theorem 4.3.2]).

The above examples motivate to ask what type of quivers admit only finitely many isomorphism classes of representations of any prescribed dimension vector. A theorem due to Gabriel yields a complete description of these quivers, their underlying undirected graphs are simply-laced Dynkin diagrams. For a more detailed account of this theorem, see [6, Sec. 4.7], [9, Sec. 2.4].

The path algebra

Let Q be a quiver. A path p in a quiver Q is a finite sequence $(a_1, a_2, \ldots, a_n)$ of arrows such that $t(a_i) = s(a_{i+1})$ for $i = 1, 2, \ldots, n-1$.

For a path $p = (a_1, a_2, \ldots, a_n)$, we define $s(p) := s(a_1)$ and $t(p) := t(a_n)$. For every vertex $v \in Q_0$, we define a trivial path e_v with $s(e_v) = t(e_v) = v$.

If $p = (a_1, a_2, \ldots, a_n)$ and $q = (b_1, b_2, \ldots, b_m)$ are two paths such that $t(p) = s(q)$, then we can define the path pq by

$$pq := (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m).$$

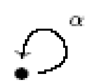
We set $e_{s(p)}p = p$ and $pe_{t(p)} = p$.

Definition 1.5. The associative algebra kQ is the k-vector space spanned by all paths (including trivial paths) in Q with the multiplication defined as follows:

$$p \cdot q = \begin{cases} pq & \text{if} t(p) = s(q) \\ 0 & \text{otherwise} \end{cases}$$

This associative algebra kQ is called the path algebra of a quiver Q over a field k.

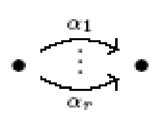
Example 1.6. Consider the loop quiver

$L :$ 

Then, it is clear that the path algebra of L is the polynomial ring $k[T]$ in one variable with coefficients in k.

More generally, the path algebra of the r-loop L_r is the non-commutative free algebra $k\langle X_1, \ldots, X_r\rangle$ with r generators.

Example 1.7. Consider the r-arrow Kronecker quiver

$K_r :$ 

The path-algebra of the r-arrow Kronecker quiver K_r has basis

$e_i, e_j, \alpha_1, \ldots, \alpha_r$. Thus, kK_r is the direct sum of $k\alpha_1 \oplus \cdots \oplus k\alpha_r$, with $ke_i \oplus ke_j$ (a subalgebra isomorphic to $k \times k$).

In the form of matrix algebra, we can write it as

$$kK_r = \begin{pmatrix} k & H \\ \\ 0 & k \end{pmatrix}$$

where H is an r-dimensional k-vector space. Therefore, we sometime represent the r-arrow Kronecker quiver by simply

$$\bullet \xrightarrow{H} \bullet$$

where H is the multiplicity space of arrows having dimension r as k-vector space.

Let **Mod**-kQ denote the category of right kQ-modules. It is well known that the categories **Mod**-kQ and $\mathbf{Rep}_k(Q)$ are equivalent [4, Theorem 1.5]. Therefore, in the following, we use the same notation for a module and the corresponding representation.

Given $v \in Q_0$, consider the representation $S(v)$ defined by

$$S(v)_v = k, \quad S(v)_w = 0 (w \in Q_0, v \neq w), \quad S(v)_a = 0 (a \in Q_1).$$

Then clearly, the representation $S(v)$ is simple with dimension vector $\epsilon_v = \Big((\epsilon_v)_w\Big)$, where $(\epsilon_v)_w = 1$, if $v = w$ and $(\epsilon_v)_w = 0$, otherwise. If kQ is finite dimensional, then all simple representations are of the above type only [9, Proposition 1.3.1].

We say that Q has no oriented cycles if there is no non-trivial path p such that $s(p) = t(p)$. It is clear from the definition of path algebra that the algebra kQ is a finite dimensional k-vector space if and only if Q has no oriented cycles.

1.1. The geometric approach

We fix a quiver Q and a dimension vector $\alpha \in \mathbf{Z}_{\geq 0}^{Q_0}$. Choose vector spaces M_v of dimension α_v. Then the isomorphism classes of representations of Q with dimension vector α are in natural one-to-one correspondence with the orbits in the representation space

$$\mathcal{R}(Q, \alpha) := \bigoplus_{a \in Q_1} Hom_k(M_{s(a)}, M_{t(a)})$$

of the reductive group

$$GL(\alpha) := \prod_{v\in Q_0} GL(M_v)$$

acting by

$$(g\cdot\phi)_a := g_{t(a)}\phi_a g^{-1}_{s(a)},$$

where $g = (g_v)_{v\in Q_0} \in GL(\alpha)$ and $\phi = (\phi_a)_{a\in Q_1} \in \mathcal{R}(Q,\alpha)$.

Note that the one-parameter subgroup

$$\Delta := \{(tId_{M_v})_{v\in Q_0} | t \in k^*\}$$

of $GL(\alpha)$ acts trivially on $\mathcal{R}(Q,\alpha)$. Thus, the action of $GL(\alpha)$ factors through an action of the quotient group

$$PGL(\alpha) := GL(\alpha)/\Delta\,.$$

It is clear that any point $x \in \mathcal{R}(Q,\alpha)$ defines a representation M_x of Q, and vice versa. Moreover, two points x and y are in the same orbit of $GL(\alpha)$ (or, equivalently of $PGL(\alpha)$) if and only if the corresponding representations M_x and M_y are isomorphic.

Given a representation M of Q with dimension vector α, we denote by O_M, the orbit of the corresponding point in $\mathcal{R}(Q,\alpha)$ with respect to the natural action of $GL(\alpha)$ on $\mathcal{R}(Q,\alpha)$. Then,

$$O_M = \{x \in \mathcal{R}(Q,\alpha) \,|\, M_x \cong M\}$$

Proposition 1.8. *[9, Theorem 2.3.1] Let*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \tag{1.1}$$

be an exact sequence of finite dimensional representations of Q.

1. *If the sequence (1.1) is non-split, then $O_{M'\oplus M"} \subset \overline{O_M} \setminus O_M$.*
2. *The exact sequence (1.1) splits if and only if $O_M = O_{M'\oplus M''}$.*

As a consequence of the Proposition 1.8 and the Jordan-Hölder theorem (see [6, Theorem 1.1.4]), we have

Corollary 1.9. *The orbit O_M is closed if and only if the representation M is semi-simple.*

Now, consider the affine quotient of $\mathcal{R}(Q,\alpha)$ by $GL(\alpha)$, that is,

$$\mathcal{R}\left(Q,\alpha\right)//GL\left(\alpha\right) := Spec\left(I\left(Q,\alpha\right)\right),$$

where $I(Q,\alpha) := k[\mathcal{R}(Q,\alpha)]^{GL(\alpha)}$ is the ring of $GL(\alpha)$-invariant polynomials on $\mathcal{R}(Q,\alpha)$. By standard results from geometric invariant theory, it follows that there is a natural bijection from the set of closed points of the affine algebraic variety $Spec(I(Q,\alpha))$ to the set of closed $GL(\alpha)$-orbits in $\mathcal{R}(Q,\alpha)$. By Corollary 1.9, it follows that the affine quotient $\mathcal{R}\left(Q,\alpha\right)//GL\left(\alpha\right)$ parametrizes semi-simple representations of dimension vector α.

Remark 1.10. If Q has no oriented cycle, then any semi-simple representation with dimension vector α is isomorphic to $\bigoplus_{v\in Q_0}\alpha_v S(v)$, see Section 1. Therefore, origin is the only point in $\mathcal{R}(Q,\alpha)$ which corresponds to semi-simple representations, and hence every orbit closure contains the origin. Hence, the moduli spaces provided by the classical invariant theory is not interesting in this case. This is because the classical theory only picks out closed $GL(\alpha)$-orbits in $\mathcal{R}(Q,\alpha)$, which correspond to semi-simple representations. To get interesting moduli spaces of representations, A. King has applied Mumford's geometric invariant theory, with trivial linearization twisted by a character χ of $GL(\alpha)$. We will discuss this approach in §3.

2. Derksen-Weyman-Schofield semi-invariants

In this section, we will give an outline of the main results of [13] and [40].

Given a vertex $v \in Q_0$, one can define the standard indecomposable representation $P(v)$ by $e_v kQ$. Note that $P(v)_w = e_v kQ e_w$ is the vector space spanned by all paths from v to w. Using the decomposition $1 = \sum_{v\in Q_0} e_v$, we obtain the corresponding decomposition

$$kQ \simeq \bigoplus_{v\in Q_0} P(v)\,.$$

For any representation M of Q, the evaluation map

$$\varepsilon_M\colon Hom_Q(P(v),M) \longrightarrow M_v$$

given by $f \mapsto f_v(e_v)$ is a natural isomorphism. From this it follows that $P(v)(v\in Q_0)$ are projective kQ-modules.

Proposition 2.1. *[9, Proposition 1.4.1] For any (right) kQ-module M, we have an exact sequence of kQ-modules*

$$0 \longrightarrow \bigoplus_{a\in Q_1} Me_{s(a)}\otimes_k P(t(a)) \xrightarrow{\xi} \bigoplus_{v\in Q_0} M_{e_v}\otimes_k P(v) \xrightarrow{\eta} M \longrightarrow 0\,. \tag{2.1}$$

The exact sequence eq-Ringel is called the standard (Ringel) resolution of the kQ-module M; it is a projective resolution of length at most 1.

Given any two kQ-modules M and N, consider a projective resolution

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0.$$

By applying the functor $Hom_Q(-,N)$, we obtain a complex

$$Hom_Q(P_0,N) \longrightarrow Hom_Q(P_1,N) \longrightarrow Hom_Q(P_2,N) \longrightarrow \cdots$$

The i^{th} homology group is denoted by $Ext^i_Q(M,N)$, which is independent of the choice of a projective resolution of M (see, [6, §2.4]). The space $Ext^1_Q(M,N)$ is called the space of self-extensions of M.

Using the standard resolution, one obtain the following:

Corollary 2.2. *For any representations M and N of a quiver Q, the map*

$$d^M_N\colon \prod_{v\in Q_0} Hom_k(M_v,N_v) \longrightarrow \prod_{a\in Q_1} Hom_k(M_{s(a)},N_{t(a)})$$

given by

$$(u_v)_{v\in Q_0} \mapsto (u_{t(a)}M_a - N_a u_{s(a)})_{a\in Q_1}$$

has kernel $Hom_Q(M,N)$ and cokernel $Ext^1_Q(M,N)$. Moreover, $Ext^j_Q(M,N) = 0$ for all $j\geq 2$.

By considering the dimensions in the Corollary 2.2, we have

$$\dim Hom_Q(M,N) - \dim Ext^1_Q(M,N) = \sum_{v\in Q_0}\alpha_v\beta_v - \sum_{a\in Q_1}\alpha_{s(a)}\beta_{t(a)}, \tag{2.2}$$

where α and β are dimension vectors of M and N respectively. From this, it follows that the left-hand side of eq-dim-hom-ext depends only on the dimension vectors of M,N and is a bi-additive functions of these vectors. This motivates the following:

If $\alpha,\beta\in \mathbf{Z}^{Q_0}$, then we define the Euler form by

$$\langle\alpha,\beta\rangle = \sum_{v\in Q_0}\alpha_v\beta_v - \sum_{a\in Q_1}\alpha_{s(a)}\beta_{t(a)}. \tag{2.3}$$

The ring of semi-invariants

We will now apply invariant theory to quiver representations. Let α be a dimension vector of a quiver Q. We have the algebraic group $GL(\alpha)$ acting on the representation space $\mathcal{R}(Q,\alpha)$. Recall that the ring of invariants of $\mathcal{R}(Q,\alpha)$ defined as

$$I(Q,\alpha) := k[\mathcal{R}(Q,\alpha)]^{GL(\alpha)}$$

is trivial if Q has no oriented cycles. In this case, we do not get an interesting object by considering the affine quotient. Therefore, we will look at the ring of semi-invariants.

Let $\chi\colon GL(\alpha) \longrightarrow k^*$ be a character. Such a character always looks like

$$g \mapsto \prod_{v\in Q_0} \det(g_v)^{\theta_v}$$

where $\theta \in \Gamma := \mathbf{Z}^{Q_0}$ is called the weight. The character defined by the function $\theta\colon Q_0 \longrightarrow \mathbf{Z}$ will be denoted by χ_θ. We can view weights as dual to dimension vectors $Hom_{\mathbf{Z}}(\Gamma,\mathbf{Z})$ as follows: If α is a dimension vector, then we define

$$\theta(\alpha) = \sum_{v\in Q_0} \theta_v \alpha_v \,.$$

Conversely, if $\sigma \in Hom_{\mathbf{Z}}(\Gamma,\mathbf{Z})$, then we define

$$\theta_v = \sigma(\varepsilon(v)),$$

where $\varepsilon(v)$ is the dimension vector given by $\varepsilon(v)_v = 1$ and $\varepsilon(v)_w = 0$, if $v \neq w$.

Recall that a function $f \in k[\mathcal{R}(Q,\alpha)]$ is called semi-invariant of weight χ_θ if

$$f(g\cdot x) = \chi_\theta(g) f(x)\,.$$

The ring of semi-invariants is defined as

$$SI(Q,\alpha) := k[\mathcal{R}(Q,\alpha)]^{SL(\alpha)} = \bigoplus_{\theta\in\Gamma} SI(Q,\alpha)_{\chi_\theta}\,.$$

where $SI(Q,\alpha)_{\chi_\theta}$ is the space of all semi-invariants of weight χ_θ. Note that if $SI(Q,\alpha)_{\chi_\theta} \neq \emptyset$, then $\theta(\alpha) = 0$. To see this, consider the action of $tId \in GL(\alpha)$. If f is a semi-invariant of weight χ_θ, then

$$f = \prod_{v\in Q_0} \det(tId_{\theta_v})^{\theta_v} f = \prod_{v\in Q_0} t^{\theta(\alpha)} f\,.$$

Hence, $\theta(\alpha) = 0$.

Let M and N be two representations of a quiver Q of dimension vector α and β respectively. Let us assume that the Euler form (see Euler-form)

$$\langle \alpha, \beta \rangle = 0 \,.$$

Then the linear map

$$d_N^M \colon \prod_{v \in Q_0} Hom_k(M_v, N_v) \longrightarrow \prod_{a \in Q_1} Hom_k(M_{s(a)}, N_{t(a)})$$

is represented by a square matrix. We now define $c(M, N) := \det(d_N^M)$. For fixed $M \in \mathbf{Rep}_k(Q, \alpha)$, we get a semi-invariant $c^M := c(M, -) \in SI(Q, \beta)$. Similarly, for a fixed representation N of Q of dimension vector β, we get semi-invariant $c_N := c(-, N) \in SI(Q, \alpha)$. This is well-defined up to a scalar which will be fixed once chosen a basis for the vector spaces M_v and N_v for all $v \in Q_0$. The semi-invariants c^M and c_N are called Schofield semi-invariants. One can easily check that c^M has weight χ_θ, where

$$\theta_v = \langle \alpha, \varepsilon(v) \rangle \,.$$

Recall that for any dimension vector γ, we defined

$$\theta(\gamma) = \sum_{v \in Q_0} \theta_v \gamma_v \,.$$

In this way, we can view θ as a function on dimension vectors, and hence, the weights and dimension vectors can be viewed as being dual to each other. Now we have $\theta(\varepsilon(v)) = \theta_v = \langle \alpha, \varepsilon(v) \rangle$. In particular, we can write $\theta = \langle \alpha, - \rangle$. Therefore, the semi-invariant c^M lies in $SI(Q, \beta)_{\langle \alpha, - \rangle}$ and the semi-invariant c_N lies in $SI(Q, \alpha)_{\langle -, \beta \rangle}$.

Theorem 2.3. *[13, 40] The ring $SI(Q, \beta)$ is a k-linear span of the semi-invariants c^M, where M runs through all representations of Q with $\langle \underline{\dim}(M), \beta \rangle = 0$. In particular, $SI(Q, \beta)_{\chi_\theta}$ is spanned by c^M's, where $M \in \mathbf{Rep}_k(Q, \alpha)$ and $\theta = \langle \alpha, - \rangle$ with $\theta(\beta) = 0$.*

3. King's approach

We will give a brief outline of the moduli construction of quiver representations given by A. King [18]. D. Mumford has developed the geometric invariant theory to construct the moduli spaces of geometric objects in algebraic geometry (see [25, 27, 30] for detailed account of this theory). Here, we mainly follow the presentation of [18]. Now onwards, we assume that k is an algebraically closed field; unless otherwise mentioned.

Stability of quiver representations

Fix $\theta \in \Gamma$. For a representation M of a quiver Q, we define

$$\theta(M) := \sum_{v \in Q_0} \theta_v \dim M_v \,.$$

Definition 3.1. We say that a representation M of a quiver Q is θ-semistable if $\theta(M) = 0$ and every subrepresentation $M' \subseteq M$ satisfies $\theta(M') \geq 0$. We say that M is θ-stable if the only subrepresentations M' with $\theta(M') = 0$ are M and 0.

For a general notion of semistability in an abelian category, see [33]. A more detailed account of the algebraic aspect of stability of quiver representations can be found in [32, §4].

Let us fix a dimension vector α. Let χ_θ be a character of $GL(\alpha)$ corresponding to θ.

Definition 3.2. We say that a point $x \in \mathcal{R}(Q, \alpha)$ is χ_θ-semistable if there exists a semi-invariant $f \in SI(Q, \alpha)_{\chi_\theta^n}$ with $n \geq 1$ such that $f(x) \neq 0$. We say that a point $x \in \mathcal{R}(Q, \alpha)$ is χ_θ-stable if x is χ_θ-semistable and, further $\dim G \cdot x = \dim PGL(\alpha)$ and the $GL(\alpha)$-action on $\{f \in k[\mathcal{R}(Q, \alpha)] \mid f(x) \neq 0\}$ is closed.

We shall denote by $\mathcal{R}(Q, \alpha)^{ss}$ the set of θ-semistable points, and by $\mathcal{R}(Q, \alpha)^s$ the set of θ-stable points in $\mathcal{R}(Q, \alpha)$.

The GIT quotient of $\mathcal{R}(Q, \alpha)$ by $GL(\alpha)$ with respect to a linearization given by the character χ_θ is

$$M_Q^{ss}(\alpha, \theta) := Proj\left(\bigoplus_{n \geq 0} SI(Q, \alpha)_{\chi_\theta^n} \right).$$

We have a natural morphism $\pi \colon \mathcal{R}(Q, \alpha)^{ss} \longrightarrow M_Q^{ss}(\alpha, \theta)$ which is a good categorical quotient. In fact, the GIT quotient $M_Q^{ss}(\alpha, \theta)$ can be described in terms of equivalence classes of orbits as follows:

Two points $x, y \in \mathcal{R}(Q, \alpha)^{ss}$ determine the same point in $M_Q^{ss}(\alpha, \theta)$ if and only if

$$\overline{O_{M_x}} \cap \overline{O_{M_y}} \cap \mathcal{R}(Q, \alpha)^{ss} \neq \emptyset \,.$$

Since the closure of every orbit contains a unique closed orbit, it follows that $M_Q^{ss}(\alpha, \theta)$ parametrizes the closed orbits in $\mathcal{R}(Q, \alpha)^{ss}$. Note that here we are taking orbits that are closed in $\mathcal{R}(Q, \alpha)^{ss}$, not in $\mathcal{R}(Q, \alpha)$.

By general theory, we get a projective morphism

$$p\colon M_Q^{ss}(\alpha,\theta) \longrightarrow \mathcal{R}\,(Q,\alpha)\,//GL\,(\alpha)$$

which sends every semistable orbit O_M to the unique closed orbit contained in $\overline{O_M}$.

If Q is a quiver without oriented cycles, then $\mathcal{R}\,(Q,\alpha)\,//GL\,(\alpha)$ is a point. Hence, in this case, the GIT quotient $M_Q^{ss}(\alpha,\theta)$ is a projective variety.

We define

$$M_Q^{s}(\alpha,\theta) := \pi(\mathcal{R}(Q,\alpha)^s).$$

The algebraic semistability notion (see Definition 3.1) for representations of a quiver is indeed a GIT-notion. For this, we have

Proposition 3.3. *[18, Proposition 3.1] A point $x \in \mathcal{R}(Q,\alpha)$ is χ_θ-semistable (respectively χ_θ-stable) if and only if the corresponding representation M_x is θ-semistable (respectively θ-stable).*

Let us note that for any morphism between θ-semistable representations, the kernel, image and cokernel of the morphism are all θ-semistable and hence the θ-semistable representations form an abelian subcategory of kQ-**Mod**. Moreover, the simple objects in this subcategory are precisely the θ-stable representations. Since this category is Noetherian and Artinian, the Jordan-Hölder theorem holds, and so we have a notion of S-equivalence in this category. More precisely, given a θ-semistable representation M, there exists a Jordan-Hölder filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M,$$

such that all quotients M_i/M_{i-1} are θ-stable representations. By the Jordan-Hölder theorem, the representation

$$gr(M) := \bigoplus_{i=1}^{r} M_i/M_{i-1}$$

depends only on M. The representation $gr(M)$ is called the associated graded representation to M in the category of θ-semistable representations.

Definition 3.4. We say that two θ-semistable representations are S-equivalent if they have isomorphic associated graded representations in the category of θ-semistable representations. We say that a representation M of Q is θ-polystable if M is direct sum of θ-stable representations.

Proposition 3.5. *[18, Proposition 3.2] Let M and N be two θ-semistable representations of a quiver Q. Then*

(1) *A $GL(\alpha)$-orbit O_M of a θ-semistable representation M is closed in $\mathcal{R}(Q,\alpha)^{ss}$ if and only if M is θ-polystable.*

(2) *$\overline{O_M} \cap \overline{O_N} \cap \mathcal{R}(Q,\alpha)^{ss} \neq \emptyset$ if and only if M and N are S-equivalent.*

Using Propositions 3.3, 3.5, we have a more precise description of the varieties $M_Q^{ss}(\alpha,\theta)$ and $M_Q^{s}(\alpha,\theta)$ in terms of algebraic stability of quiver representations (see, Definition 3.1). More precisely, the closed points of quotient $M_Q^{ss}(\alpha,\theta)$ are in one-to-one correspondence with S-equivalence classes of θ-semistable representations of Q. The closed points of an open subset $M_Q^{s}(\alpha,\theta)$ of $M_Q^{ss}(\alpha,\theta)$ correspond to the isomorphism classes of θ-stable representations of Q.

Moduli of representations

We will now describe that the variety $M_Q^{ss}(\alpha,\theta)$ 'corepresents' the appropriate moduli functor which justifies the use of the term 'moduli space' for this variety. In the terminology introduced by Simpson [39, Section 1], a (coarse) moduli space is a scheme which corepresents a moduli functor. We first recall some basic definitions. Let $\mathbf{Sch}^\circ$ be the opposite category of schemes and $\mathbf{Set}$ be the category of sets. For a scheme Z, its functor of points

$$\underline{Z}\colon \mathbf{Sch}^\circ \longrightarrow \mathbf{Set}$$

is given by $X \mapsto Hom(X,Z)$. By Yoneda Lemma, every natural transformation $\underline{Y} \longrightarrow \underline{Z}$ is of the form $\underline{f}$ for some morphism of schemes $f\colon Y \longrightarrow Z$.

Definition 3.6. Let $\mathbf{M}\colon \mathbf{Sch}^\circ \longrightarrow \mathbf{Set}$ be a functor, $\mathcal{M}$ a scheme and $\psi\colon \mathbf{M} \longrightarrow \underline{\mathcal{M}}$ a natural transformation. We say that $\mathcal{M}$ corepresents $\mathbf{M}$ if for each scheme Y and each natural transformation $h\colon \mathbf{M} \longrightarrow \underline{Y}$, there exists a unique $g\colon \mathcal{M} \longrightarrow Y$ such that $h = \underline{g} \circ \psi$, that is the following diagram

$$\begin{array}{ccc} \mathbf{M} & & \\ \psi \downarrow & \searrow^{h} & \\ \underline{\mathcal{M}} & \xrightarrow[\underline{g}]{} & \underline{Y} \end{array}$$

commutes.

A family of kQ-modules over a connected scheme S is a locally-free sheaf $\mathcal{F}$ over S together with a k-algebra homomorphism $kQ \longrightarrow End(\mathcal{F})$. On the other hand, a family of representations of Q is a representation of Q in the category of locally-free sheaves over S. The equivalence between the kQ-modules and the representations of Q extends naturally to families.

Let us consider the moduli functor

$$\mathcal{M}_Q^{ss} := \mathcal{M}_Q^{ss}(\alpha, \theta) \colon \mathbf{Sch}^\circ \longrightarrow \mathbf{Set}$$

where $\mathcal{M}s_Q^{ss}(S)$ is the set of all isomorphism classes of families over S of θ-semistable representations with dimension vector α.

Theorem 3.7. *[18, Proposition 5.2] The variety $M_Q^{ss}(\alpha, \theta)$ corepresents the moduli functor $\mathcal{M}_Q^{ss}$, and the closed points of $M_Q^{ss}(\alpha, \theta)$ correspond to the S-equivalence classes of θ-semistable representations of Q.*

We say that a dimension vector α is a Schur root if there exists a representation of dimension vector α with trivial endomorphism ring. The precise criterion for the existence of θ-stable representation of a quiver Q having dimension vector α is given by the following result (see, [34]).

Proposition 3.8. *[18, Proposition 4.4] There exists some $\theta \in \Gamma$ for which $M_Q^s(\alpha, \theta)$ is non-empty if and only if α is a Schur root.*

We say that some property is true for a general representation with dimension vector α if this property is independent of the point chosen in some non-empty open subset of $\mathcal{R}(Q, \alpha)$. A dimension vector β is called a general subvector of α if a general representation of dimension vector α has a subrepresentation of dimension vector β. In general, it is not possible to choose θ such that general representations of all Schur roots are θ-stable. However, the following result gives a more stronger characterization of Schur roots.

Theorem 3.9. *[34, Theorem 6.1] Let α be a dimension vector for the quiver Q, and let $\theta_\alpha(\beta) := \langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle$, where $\langle , \rangle$ is the Euler form Euler-form. Then, the dimension vector α is Schur root if and only if the general representation of dimension vector α is θ_α-stable.*

The notion of general representation and general subvector are quite useful in studying the birational classification of $M_Q^{ss}(\alpha, \theta)$ (see [34, 35] for more details).

Remark 3.10. The path algebra of a (finite) quiver is hereditary, i.e., every submodule of a projective module is projective. In fact, over an algebraically closed field, every finite dimensional hereditary algebra is Morita equivalent to the path algebra of a (finite) quiver without oriented cycles. Thus, the representations of quivers are quite useful in order to study the representations of finite dimensional k-algebras (see [6, Chapter 4]). Let k be an algebraically closed field. Given any finite dimensional k-algebra Λ, there exist a quiver Q (called the quiver associated to Λ) such that the category $\mathbf{Mod}(\Lambda)$ is an abelian subcategory of kQ-$\mathbf{Mod}$. The vertices of Q are in one-one correspondence with the isomorphism classes of simple Λ-modules, and thus the Grothendieck group $K_0(\mathbf{Mod}(\Lambda))$ is the free abelian group generated by Q_0. Therefore, an integer-valued function θ on $\mathbf{Mod}(\Lambda)$ is merely an element of $\mathbf{Z}^{Q_0}$, and hence determines a character of $GL(\alpha)$ as discussed in Section 1.1. For a more precise account of moduli of representations of a finite dimensional algebra, see [18, Section 4].

For further study of moduli of representations, one may consult [10, 12, 31, 35]. This list is very far from being complete, it should rather be considered as starting points for further reading.

4. Applications

In this section, we shall briefly outline the functorial approach to construct the moduli of sheaves using moduli of representations of a Kronecker quiver [1].

Brief review of the constructions

D. Mumford [28] introduced the notion of semistability for vector bundles on a curve, and constructed the moduli space of stable vector bundles over a curve as a quasi-projective variety, by reducing the problem in GIT. Latter, C. S. Seshadri [36] introduced the notion of S-equivalence for semistable vector bundles, and constructed the moduli space of semistable vector bundles to obtain a compactification of the moduli space of stable vector bundles over a curve using GIT. These concepts and techniques have been extended to torsion-free sheaves on higher dimensional varieties [17, 21, 22]. The advantage of restricting to semistable sheaves is that the moduli problem is bounded. To construct a moduli space using GIT, one need to have a boundedness result, and this is one of the complications in extending the techniques to higher dimensional varieties. C. T. Simpson [39] extended the notion of semistability to 'pure' sheaves, and using

Maruyama's boundedness result, he obtained the boundedness of semistable sheaves over a projective scheme of finite type defined over an algebraically closed field of characteristic zero. When the characteristic of the base field is positive, the boundedness result for semistable sheaves is obtained by A. Langer [20].

Let us recall the basic steps in the construction of moduli of semistable sheaves with a fixed Hilbert polynomial P.

- The first step is to identify the isomorphism classes of semistable sheaves with Hilbert polynomial P with orbits in certain Quot scheme for certain action of a reductive group.
- The second step is to find a projective embedding of Quot scheme to reduce the problem in to the GIT quotient.

C. Simpson used Grothendieck's embedding of Quot scheme into a Grassmannian, and constructed the moduli space of semistable sheaves as a GIT quotient. In [1], Álvarez-Cónsul and King have combined these two steps and embedded the moduli functor of semistable sheaves into the moduli functor of representations of a Kronecker quiver. A substantial part of the work in [1] is to relate the notion of stability of sheaves with the stability of the corresponding representations of this quiver. The problem of constructing moduli space of semistable sheaves then reduced to construct the GIT quotient of certain locally closed subscheme of a representation space of this quiver. In the following, we will give a brief overview of this functorial construction of moduli of semistable sheaves over a projective variety.

Sheaves and Kronecker modules

Let $(X, \mathcal{O}_X(1))$ be a polarized projective variety over an algebraically closed field k. By a sheaf on X, we mean a coherent sheaf of $\mathcal{O}_X$-module. Let $\mathcal{E}$ be a non-zero sheaf on X. Its dimension is the dimension of the support $Supp(\mathcal{E}) := \{x \in X \mid \mathcal{E}_x = 0\} \subset X$.

We say that $\mathcal{E}$ is *pure* if the dimension of any non-zero subsheaf $\mathcal{F} \subset \mathcal{E}$ equals the dimension of $\mathcal{E}$. The Hilbert polynomial $P(\mathcal{E})$ is given by

$$P(\mathcal{E}, \ell) = \sum_{i=0}^{\infty} (-1)^i h^i(\mathcal{E}(\ell)),$$

where $h^i(\mathcal{E}(\ell)) = \dim H^i(X, \mathcal{E}(\ell))$. It is known that

$$P(\mathcal{E}, \ell) = r\ell^d/d! + \text{terms of lower degree in } \ell,$$

where d is the dimension of $\mathcal{E}$ and $r = r(\mathcal{E})$ is a positive integer. Let us denote by $\mathbf{Coh}(X)$ the category of sheaves on X.

Definition 4.1. A sheaf $\mathcal{E}$ on X is called semistable if $\mathcal{E}$ is pure and, for each nonzero subsheaf $\mathcal{E}' \subset \mathcal{E}$,

$$\frac{P(\mathcal{E}')}{r(\mathcal{E}')} \leq \frac{P(\mathcal{E})}{r(\mathcal{E})}. \tag{4.1}$$

We say that $\mathcal{E}$ is stable if the above inequality is strict for all proper subsheaves $\mathcal{E}'$ of $\mathcal{E}$.

For integers $m > n$, consider the sheaf $T = \mathcal{O}_X(-n) \oplus \mathcal{O}_X(-m)$ and a finite dimensional k-algebra

$$A = \begin{pmatrix} k & H \\ 0 & k \end{pmatrix}$$

It is clear that A is the path algebra of the Kronecker quiver $K : \bullet \xrightarrow{H} \bullet$, where $H = H^0(\mathcal{O}_X(m-n))$ is the multiplicity space for arrows. Note that a right A-module is a pair (V, W) of finite dimensional k-vector spaces together with a Kronecker module map $\alpha : V \otimes_k H \longrightarrow W$. Let us denote by $\mathbf{Mod}$-A, the category of (right) A-modules. There is an intrinsic notion of semistability for A-modules.

Definition 4.2. An A-module $M = V \oplus W$ is called semistable if for each non-zero submodule $M' = V' \oplus W'$ of M,

$$\frac{\dim V'}{\dim W'} \leq \frac{\dim V}{\dim W}. \tag{4.2}$$

We say that M is stable if the above inequality is strict for all proper submodules M' of M.

The above notion of semistability for A-modules coincide with the θ-semistability of representations of Kronecker quiver for $\theta = (-b, a)$. Therefore, by King's general construction of moduli of representations of a quiver, one obtains the moduli of semistable A-modules, which we denote by $M_A^{ss}(a, b) := M_K^{ss}(\alpha, \theta)$, where $\alpha = (a, b)$ and $\theta = (-b, a)$.

For any sheaf $\mathcal{E}$ on X, we have the quiver representation

$$Hom_X(T, E) = H^0(\mathcal{E}(n)) \oplus H^0(\mathcal{E}(m))$$

together with $\alpha_{\mathcal{E}}: H^0(\mathcal{E}(n)) \otimes H \to H^0(\mathcal{E}(m))$. We obtained a functor

$$\Phi := Hom_X(T, -): \mathbf{Coh}(X) \longrightarrow \mathbf{Mod}\text{-}A$$

given by $\mathcal{E} \mapsto Hom_X(T, E)$. The functor Φ has a left adjoint

$$\Phi^{\vee} := - \otimes_A T : \mathbf{Mod}\text{-}A \longrightarrow \mathbf{Coh}(X)$$

which is quite useful in describing the functorial embedding of semistable sheaves in the category of Kronecker modules. More precisely, if $\mathcal{O}_X(m-n)$ is regular, then the functor Φ induces a fully faithful embedding of the full subcategory of n-regular sheaves in the category of Kronecker modules [1, Theorem 3.4].

A functorial construction

Fix a Hilbert polynomial P. Let us consider the moduli functor $\mathcal{M}_X(P): \mathbf{Sch}^{\circ} \longrightarrow \mathbf{Set}$ which assigns to each scheme S the set of all isomorphism classes of flat families over S of n-regular sheaves on X having Hilbert polynomial P. There are open sub-functors $\mathcal{M}_X^s \subset \mathcal{M}_X^{ss}$ of $\mathcal{M}_X$ defined by demanding that all the sheaves in flat families are stable or semistable, respectively. There are also sub-functors $\mathcal{M}_X^{reg}(n) \subset \mathcal{M}_X$ of n-regular sheaves, for any fixed integer n. The main ingredient in the construction is the careful analysis of the preservation of semistability under the functor Φ.

Theorem 4.3. *[1, Theorem 5.10, Corollary 5.11] For sufficiently large $m \gg n \gg 0$, the following holds:*

1. *A sheaf $\mathcal{E}$ on X is semistable if and only if it is pure, n-regular and the A-module $\Phi(\mathcal{E})$ is semistable.*
2. *The functor Φ preserves the S-equivalence classes, i.e., the semistable sheave $\mathcal{E}$ and $\mathcal{E}'$ having Hilbert polynomial P are S-equivalent if and only if the A-modules $\Phi(\mathcal{E})$ and $\Phi(\mathcal{E}')$ are S-equivalent.*

Therefore, the functor Φ gives a set-theoretic embedding of S-equivalence classes of semistable sheaves with Hilbert polynomial P into S-equivalence classes of semistable Kronecker modules with dimension vector $(P(n), P(m))$.

By fixing the vector spaces V and W of dimension $P(n)$ and $P(m)$ respectively, we obtain the linear space $R = Hom(V \otimes H, W)$ parametrizing

Kronecker modules of dimension vector $(P(n), P(m))$, with linear action of the reductive group $G = GL(V) \times GL(W)$.

An important key point is that the Kronecker module M is in the image of the functor Φ is locally closed condition in R ([1, Proposition 4.2]). This enables to prove that the moduli functor $\mathcal{M}_X^{reg}(n)$ is locally isomorphic to the quotient functor $\underline{Q}/\underline{G}$, where Q is a locally closed G-invariant subscheme of R (see, [1, Theorem 4.5]). This provides a key ingredient in the construction of moduli space of semistable sheaves on X with Hilbert polynomial P; essentially replacing the Quot scheme in the usual construction. The open subscheme $Q^{[ss]}$ of Q parametrizing semistable sheaves is a locally closed subscheme of R^{ss}. It turn out that the moduli functor $\mathcal{M}_X^{ss}(P)$ is locally isomorphic to the quotient functor $\underline{Q^{[ss]}}/\underline{G}$. Therefore, the problem of construction of moduli scheme $M_X^{ss}(P)$ which corepresent the moduli functor $\mathcal{M}fss$ reduced to the problem of existence of a good quotient of $Q^{[ss]}$ by G. The moduli scheme $M_X^{ss}(P)$ constructed as a good quotient of $Q^{[ss]}$ by G ([1, Proposition 6.3]) is, a priori, quasi-projective. There is morphism $\varphi\colon M_X^{ss}(P) \longrightarrow M_A^{ss}(P(n), P(m))$ induced by the inclusion $Q^{[ss]} \subset R^{ss}$. If the characteristic of the filed k is zero, then φ is closed scheme-theoretic embedding . In general, we may not have the same conclusion in characteristic $p > 0$. Using the valuative criterion for properness, it is proved that $M_X^{ss}(P)$ is proper [1, Proposition 6.5].

Theorem 4.4. *[1, Theorem 6.4]*

1. *There is a projective scheme $M_X^{ss}(P)$ which corepresents the moduli functor $\mathcal{M}_X^{ss}(P)$. Moreover, the closed points of $M_X^{ss}(P)$ correspond to the S-equivalence classes of semistable sheaves with Hilbert polynomial P.*

2. *There is an open subscheme $M_X^{s}(P)$ of $M_X^{ss}(P)$ which corepresents the moduli functor $\mathcal{M}_X^{s}(P)$ and whose closed points correspond to the isomorphism classes of stable sheaves with Hilbert polynomial P.*

The determinant theta functions

It is very interesting that the functorial approach to construct the moduli spaces of sheaves is closely related to 'theta functions'. In fact, using the results of [13, 35] (see Section 2), one obtains an explicit homogeneous co-ordinates on the moduli spaces $M_A^{ss}(a, b)$ in terms of certain Schofield semi-invariants, so called determinant theta functions. Using the adjunction between Φ and $\Phi^\vee$, one obtains an explicit homogeneous co-ordinates of the moduli space $M_X^{ss}(P)$ as a restriction of determinant theta functions through the closed scheme-theoretic embedding $M_X^{ss}(P) \longrightarrow M_A^{ss}(P(n), P(m))$ (except in the case of characteristic $p > 0$).

In the following, we shall briefly outline this projective embedding of the moduli space $M_X^{ss}(P)$.

Using Theorem 2.3, one can give a characterization of the semistability of Kronecker modules in terms of invertibility of certain maps between projective modules. First note that $P_0 = Ae_0$ and $P_1 = Ae_1$ are two indecomposable projective A-modules such that $A = P_0 \oplus P_1$. If M is any A-module, then $V = Hom_A(P_0, M), \quad W = Hom_A(P_1, M)$ (see, Section 2) and the corresponding Kronecker module $\alpha\colon V \otimes H \longrightarrow W$ is given by the composition

$$Hom_A(P_0, M) \otimes H \longrightarrow Hom_A(P_1, M),$$

where $H = Hom_A(P_1, P_0)$.

A Kronecker module M of dimension vector (a, b) is semistable if and only if there is a semi-invariant f of weight $(-nb, na)$ such that $f(M) \neq 0$, for some positive integer n (see, Proposition 3.3). Using Theorem 2.3, it follows that M is semi stable if and only if there exists a Kronecker module N such that $c^N(M) \neq 0$, where c^N is a determinant semi-invariant of weight $(-nb, na)$. Note that the Kronecker module N is also semistable, and hence saturated. Hence, it follows by simple computation that N has a projective resolution of the form $\gamma\colon U_1 \otimes P_1 \longrightarrow U_0 \otimes P_0$ and $c^N = \theta_\gamma$, where $\theta_\gamma(M) := \det Hom_A(\gamma, M)$.

Let $U = (U_0, U_1)$ be a pair of finite dimensional vector spaces such that $a \dim U_0 = b \dim U_1$. Then for any flat family M over a scheme S of A-module with dimension vector (a, b), we get a line bundle $\lambda_U(M) := (\det Hom_X(U_0, V)^{-1} \otimes (\det Hom_X(U_1, W)$ over S. For any map $\gamma\colon U_1 \otimes P_1 \longrightarrow U_0 \otimes P_0$, we get a global section $\theta_\gamma(M)$ of $\lambda_U(M)$. In this way, we have a formal line bundle λ_U with a global section θ_γ on the moduli functor $\mathcal{M}_A(a, b)$. Using Kempf's descent criterion, it is proved the restriction of this formal line bundle and section to the moduli functor $\mathcal{M}_A^{ss}(a, b)$ descends to a line bundle $\lambda_U(a, b)$ and section $\theta_\gamma(a, b)$ on the moduli space $M_A^{ss}(a, b)$ [1, Proposition 7.5].

Note that the line bundle $\lambda_U(a, b)$ depends only on $\dim U_0$ and $\dim U_1$ upto isomorphism. Let $\mathbf{M}$ be the tautological family of A-modules on R. Note that $\lambda_U(\mathbf{M})$ is the G-linearized line bundle used in the construction of GIT quotient $M_A^{ss}(a, b)$. Thus, the restriction of $\lambda_U(\mathbf{M})$ to R^{ss} descends to the quotient $M_A^{ss}(a, b)$ and hence, the line bundle $\lambda_U(a, b)$ on $M_A^{ss}(a, b)$ is ample. Moreover, the space of global sections of $\lambda_U(a, b)$ is canonically isomorphic to the space of semi-invariants on R with weight χ_U [1, Proposition 7.6]. By choosing sufficiently large U_0, U_1 and using Theorem 2.3, we can find finitely many $\gamma_0, \dots, \gamma_K : U_1 \otimes P_1 \longrightarrow U_0 \otimes P_0$ such that map

$$\Theta_\gamma : M_A^{ss}(a,b) \longrightarrow \mathbf{P}^K \ ; \quad [M] \mapsto (\theta_{\gamma_0}(M) : \cdots : \theta_{\gamma_K}(M))$$

is scheme-theoretic closed embedding [1, Theorem 7.8].

Most of the above can be carried over to the case of sheaves, using the adjunction between Φ and $\Phi^\vee$. For sufficiently large enough mn, let $\mathcal{E}$ be n-regular pure sheaf of Hilbert polynomial P. Then $\mathcal{E}$ is semistable if and only if there is a map $\delta : U_1 \otimes \mathcal{O}_X(-m) \longrightarrow U_0 \otimes \mathcal{O}_X(-n)$ such that $\theta_\delta(\mathcal{E}) := \det Hom_X(\delta, \mathcal{E}) \neq 0$. Given a family $\mathcal{E}$ over a scheme S of n-regular sheaves with Hilbert polynomial P and a map $\delta : U_1 \otimes \mathcal{O}_X(-m) \longrightarrow U_0 \otimes \mathcal{O}_X(-n)$ with $P(n) \dim U_0 = P(m) \dim U_1$, we get a line bundle $\lambda_U(\mathcal{E})$ on S with a section $\theta_\delta(\mathcal{E})$. In this way, one obtains a formal line bundle on the moduli functor $\mathcal{M}_X^{ss}$ which descends to the genuine line bundle $\lambda_U(P)$ on the moduli space $M_X^{ss}(P)$. Under the embedding $\varphi : M_X^{ss}(P) \longrightarrow M_A^{ss}(P(n), P(m))$, where n, m satisfy some technical conditions (see [1, Section 5.1]), we have $\lambda_U(P) = \varphi^* \lambda_U(a,b)$ and a global section $\theta_\delta(P) = \varphi^* \lambda_U(P(n), P(m))$. Therefore, the line bundle $\lambda_U(P)$ on $M_X^{ss}(P)$ is ample [1, Proposition 7.7]. Now using the adjunction between Φ and $\Phi^\vee$, we have the following:

Theorem 4.5. *[1, Theorem 7.10] For any Hilbert polynomial P, there exist vector spaces U_0, U_1 and finitely many maps $\delta_0, \dots, \delta_K : U_1 \otimes \mathcal{O}_X(-m) \longrightarrow U_0 \otimes \mathcal{O}_X(-n)$ such that the map*

$$\Theta_\delta : M_X^{ss}(P) \longrightarrow \mathbf{P}^K \ ; \quad [\mathcal{E}] \mapsto (\theta_{\delta_0}(\mathcal{E}) : \cdots : \theta_{\delta_K}(\mathcal{E}))$$

is a closed scheme-theoretic embedding in characteristic zero. In the case of positive characteristic, it is scheme-theoretic on the stable locus $M_X^s(P)$.

4.1. Faltings theta functions

In [16], a GIT-free construction of the moduli space $M_C^{ss}(r,d)$ of semistable vector bundles of rank r and degree d on a smooth projective curve C defined over k is given by G. Faltings (see also [38]). The main ingredient is the following cohomological criterion of semistability of vector bundles on C.

Theorem 4.6. *[16, 38] A vector bundle E on C is semistable if and only if there exists a vector bundle F on C such that $E \otimes F$ is cohomologically trivial, i.e., $H^0(C, E \otimes F) = 0 = H^1(C, E \otimes F)$*

The above characterization of semistable vector bundles is very crucial to give a GIT-free construction of moduli of vector bundles over a smooth projective curve. To see this, suppose that $\mathcal{E}$ is a family of vector bundles on C parametrized by a scheme S and F is a vector bundle on C. Then, there is a 2-term complex $K^\bullet : K^0 \xrightarrow{d} K^1$ of vector bundles on S that computes the cohomology of $\mathcal{E} \otimes F$ locally over S. That is, the fibres of $Ker(d)$ and $Coker(d)$ at each $s \in S$ are isomorphic to $H^0(\mathcal{E}_s \otimes F)$ and $H^1(\mathcal{E}_s \otimes F)$, respectively. The complex $K^\bullet$ is not unique, but any other complex of vector bundles with this property must be quasi-isomorphic to $K^\bullet$. Since the determinant line bundles of two quasi-isomorphic complexes are isomorphic, we have a well-defined determinant line bundle on S associated to $\mathcal{E} \otimes F$, which we denote by $D(\mathcal{E} \otimes F)$. If $\chi(\mathcal{E}_s \otimes F) = 0$ for all $s \in S$, then the vector bundles appearing in $K^\bullet$ have the same rank, and hence there is a section θ_F canonically identified locally with $\det d$. For $s \in S$, observe that

$$\theta_F(s) \neq 0 \text{ if and only if } H^0(C, \mathcal{E}_s \otimes F) = 0 = H^1(C, \mathcal{E}_s \otimes F).$$

By Theorem 4.6, it follows that if $\theta_F(s) \neq 0$, then $\mathcal{E}_s$ is semistable.

Note that if $r_0 = rk(F)$ and $L = \det(F)$ are fixed, then line bundle $D(\mathcal{E} \otimes F)$ is independent of the choice of F [38, Lemma 2.5]. Therefore, we denote this determinant line bundle on S by simply $D(r_0, L, \mathcal{E})$. As F varies with given rank and degree such that $\chi(\mathcal{E} \otimes F) = 0$, we get sections θ_F of $D(r, L, \mathcal{E})$. These functions, so called Faltings theta functions, are used to give an implicit construction of the moduli space $M_C^{ss}(r, d)$. In the following, we give a brief outline of this construction.

GIT-free construction

Let $R(r, d)$ be a smooth quasi-projective variety, and let $\mathcal{E}$ be a family of vector bundles of rank r and degree d on C parametrized by $R(r, d)$ such that

- given a semistable vector bundle E on C of rank r and degree d there is an $q \in R(r, d)$ such that E is isomorphic to $\mathcal{E}_q$.
- $R(r, d)$ has the local universal property with respect to families of semistable vector bundles.

We refer to [36, 38] for the proof of the existence of $R(r, d)$ having the above properties. Let $R(r, d)^{ss}$ and $R(r, d)^{ss}$ be open subset of $R(r, d)$ parametrizing semistable and stable bundles in the family $\mathcal{E}$. By choosing a vector bundle F of fixed rank and determinant such that $\chi(\mathcal{E} \otimes F) = 0$, one gets the corresponding section θ_F of the line bundle $D(r, L, \mathcal{E})$ on $R(r,d)^{ss}$. From the discussion after Theorem 4.6, it follows that if for $q \in R$, $\theta_F(q) \neq 0$, then the corresponding vector bundle $\mathcal{E}_q$ is semistable. In fact, one can find finitely many vector bundles $F_1, F_2, \ldots, F_N$ of fixed rank and determinant such that the sections $\{\theta_{F_1}, \theta_{F_2}, \ldots, \theta_{F_N}\}$ have no base points on $R(r,d)^{ss}$ [38, Lemma 3.1]. In other words, it is possible to detect all semistable vector bundles of rank r and degree d on C by finitely many Faltings theta functions using the chomological characterization of semistable vector bundles . Let

$$\Theta \colon R(r,d)^{ss} \longrightarrow \mathbf{P}^N$$

be the corresponding morphism into the projective space. Let $M_C^{ss}(r,d)$ be the set of S-equivalence classes of semistable vector bundles of rank r and degree d on C. Then the image $M_1 = \Theta(R(r,d)^{ss})$ is a closed in $\mathbf{P}^N$, and hence it is a projective variety (see [38, p. 182]). Let $\phi \colon R(r,d)^{ss} \longrightarrow M_C^{ss}(r,d)$ be a set-theoretic map defined by $q \mapsto gr(\mathcal{E}_q)$. Then the map Θ factors through ϕ, i.e., the following diagram

$$\begin{array}{ccc} R\,(r,d)^{ss} & & \\ \phi \Big\downarrow & \searrow \Theta & \\ M_C^{ss}\,(r,d) & \xrightarrow[j_1]{} & M_1 \end{array}$$

commutes. Consider the normalization $\iota \colon M \longrightarrow M_1$ of M_1 in the function filed of $R(r,d)^{ss}$. Then M is normal projective variety and there is a set-theoretic bijection $j \colon M_C^{ss}(r,d) \longrightarrow M$ [38, p. 183-186]. Therefore, $M_C^{ss}(r,d)$ admits the structure of normal projective variety which corepresent the moduli functor of semistable vector bundles of rank r and degree d. This gives an implicit construction of the moduli space $M_C^{ss}(r,d)$. Let

$$\Theta_F : M_C^{ss}\,(r,d) \xrightarrow{\simeq} M \xrightarrow{L} M_1 \hookrightarrow \mathbf{P}^N$$

be the corresponding morphism, which is a normalization of the image of Θ.

C. S. Seshadri raised a question in [38, Remark 6.1] that how close this normalization being an isomorphism. In [15], it is proved that one can find Θ_F which is a scheme-theoretic embedding on the stable locus $M_C^s(r,d)$. By using the above functorial approach, it turns out that the determinant theta functions coincide with the Faltings theta functions, and hence by

Theorem 4.5, the above map Θ_F is a closed scheme-theoretic projective embedding of $M_C^{ss}(r,d)$ answering positively the above question [1] (see [1, Section 7.4] for more details).

4.2. Equivariant sheaves

In [2], the Álvarez-Cónsul and King's construction is extended to the moduli of equivariant sheaves on projective Γ-schemes, for a finite group Γ. In the first step, the problem is translated into the Γ-Kronecker modules by giving a straightforward generalisation of the results of [1]. The main new ingredient is to translate further the problem from Γ-Kronecker modules to the representations of an appropriate Kronecker-McKay quiver, which are called the Kronecker-McKay modules.

Let Γ be a finite group, and let Y be a Γ-projective scheme, of finite type over an algebraically closed field k of an arbitrary characteristic, with a very ample invertible Γ-linearized sheaf $\mathcal{O}_Y(1)$. For a technical reason, if the ground field is of positive characteristic, then it is assumed that the order of the group Γ is co-prime to the characteristic of the field k.

Let $\mathcal{F}$ be an $\mathcal{O}_Y$-module. Recall that a Γ-sheaf structure on $\mathcal{F}$ is the following data: For any $\gamma \in \Gamma$, an isomorphism $\lambda_\gamma : \mathcal{F} \longrightarrow (\gamma^{-1})^*\mathcal{F}$ of $\mathcal{O}_Y$-modules such that $\lambda_{\gamma\gamma'} = (\gamma^{-1})^*\lambda_{\gamma'} \circ \lambda_\gamma$ and $\lambda_{\mathbf{1}_\Gamma} = \mathbf{1}_{\mathcal{F}}$, where $\gamma, \gamma' \in \Gamma$ and $\mathbf{1}_\Gamma$ is the identity element of Γ.

Let $\mathcal{F}$ be a Γ-sheaf on Y. A subsheaf $\mathcal{F}' \subseteq \mathcal{F}$ is called Γ-subsheaf if $\lambda_\gamma(\mathcal{F}') \subseteq (\gamma^{-1})^*\mathcal{F}'$ for all $\gamma \in \Gamma$.

Let $\{\rho_1, \ldots, \rho_r\}$ be all the irreducible representations of Γ. We define

$$P_{\rho i}(E, l) := \sum_{j=0}^{\infty} (-1)^j \dim Hom_\Gamma\Big(\rho_i, H^j(E(\ell))\Big).$$

We denote the r-tuple of polynomials $(P_{\rho 1}(E), ..., P_{\rho r}(E))$ by $P_\rho(E)$. Fix r-tuple of polynomials $(P_1, \ldots, P_r)$, which we denote by τ_P. We say that a Γ-sheaf E is of type τ_P if $P_\rho(E) = \tau_P$.

Definition 4.7. A Γ-sheaf E is semistable if E is pure and, for each non-zero Γ-subsheaf $E' \subset E$, the inequality (4.1) holds. We say that a Γ-sheaf E is stable if the inequality (4.1) is strict for all proper Γ-subsheaf E'.

A Γ-sheaf E is semistable if and only if the underlying sheaf E is semistable in the usual sense.

[1] It should be mentioned here that much more is known in the curve case, namely the strange-duality conjecture is solved by A. Marian and D. Oprea [29].

For integers $m > n$, let $T := \mathcal{O}_Y(-n) \oplus \mathcal{O}_Y(-m)$ and $H := H^0(Y, \mathcal{O}_Y(m-n))$. Then for any Γ-sheaf $\mathcal{E}$, the space of global sections $H^0(\mathcal{E}(n))$ and $H^0(\mathcal{E}(m))$ has a natural Γ-module structure induced from the Γ-structure on $\mathcal{E}$ and $\mathcal{O}_Y(1)$. Moreover, the evaluation map $\alpha_E \colon H^0(E(n)) \otimes_k H \longrightarrow H^0(E(m))$ is Γ-equivariant, where H has a natural Γ-module structure. This motivates the following:

By a Γ-Kronecker module M, we mean a pair of finite dimensional Γ-modules (V, W) with a Γ-equivariant map $\alpha \colon V \otimes_k H \longrightarrow W$.

Let

$$A_\Gamma = \begin{pmatrix} k[\Gamma] & k[\Gamma] \otimes_k H \\ & \\ 0 & k[\Gamma] \end{pmatrix}$$

where H has a natural induced action of Γ.

It is easy to check that a right A_Γ-module structure on M is equivalent to a Γ-Kronecker module structure on M (see [2, Section 2]). Let $\mathbf{Mod}\text{-}A_\Gamma$ denote the category of Γ-Kronecker modules. Then, there is a natural functor

$$\Phi_\Gamma := Hom_Y(T, -) \colon \mathbf{Coh}_{\mathbf{\Gamma}}(Y) \longrightarrow \mathbf{Mod}\text{-}A_\Gamma\,. \tag{4.3}$$

Let M be an A_Γ-module. An A_Γ-submodule $M' \subset M$ is given by $k[\Gamma]$-subspaces $V' \subset V$ and $W' \subset W$ such that $\alpha(V' \otimes H) \subset W'$. There is an intrinsic notion of semistability for A_Γ-modules as follows [1, cf. p. 620].

An A_Γ-module $M = V \oplus W$ is semistable if, for each non-zero A_Γ-submodule $M' = V' \oplus W'$ of M, the inequality (4.2) holds. We say that an A_Γ-module M is stable if the inequality (4.2) is strict for all proper A_Γ-submodules M'.

Kronecker-McKay modules

Let V, H and W be finite dimensional representations of Γ over k. We have the following decomposition of finite dimensional representations

$$V = \bigoplus_i V_i \otimes \rho_i, \; W = \bigoplus_j W_j \otimes \rho_j \;\text{ and }\; \rho_i \otimes H = \bigoplus_j H_{ij} \otimes \rho_j,$$

where $V_i = Hom_\Gamma(\rho_i, V)$ and $W_j = Hom_\Gamma(\rho_j, W)$.

Now using the above decomposition and Schur's lemma, we get the following relation between morphisms

$$Hom_\Gamma(V \otimes H, W) = \bigoplus_{i,j} Hom(V_i \otimes H_{ij}, W_j). \tag{4.4}$$

Let $I_0 = I_1$ be the set of all irreducible representations of the group Γ.

Definition 4.8. The Kronecker-McKay quiver (associated to the Γ-module H) is the quiver with the vertex set a disjoint union $I_0 \sqcup I_1$ and the arrow set consists of edges with the multiplicity H_{ij} between two vertices $i \in I_0$ and $j \in I_1$.

The above definition of Kronecker-McKay quiver is motivated from the definition of McKay quiver of a group which are embedded in a linear group [24].

Let B_H denote the path algebra of this Kronecker-McKay quiver. The right B_H-modules is called Kronecker-McKay modules. Using the relation Mckay rel between morphisms, we get the relation between Γ-Kronecker modules and Kronecker-McKay modules. Let $\mathbf{Mod}\text{-}B_H$ be the category of all finitely generated Kronecker-McKay modules. Then, we get a functor

$$\Psi\colon \mathbf{Mod}\text{-}A_\Gamma \longrightarrow \mathbf{Mod}\text{-}B_H \tag{4.5}$$

which is an equivalence, in fact, it is Morita equivalence [2, Section 4.1].

A functorial embedding and preservation of semistability

Now fix a type τ_P. Let E be an n-regular Γ-sheaf of type τ_P. Then $\Phi_\Gamma(E)$ is an A_Γ-module given by $(H^0(E(n)), H^0(E(m)), \alpha_E)$. The Γ-module structure on $H^0(E(n))$ is determined by the multiplicities of irreducible representations in $H^0(E(n))$. Since E is of type τ_P, the multiplicity of ρ_i in $H^0(E(n))$ is $a_i := P_i(n),\ i = 1, \ldots, r$ and the multiplicity of ρ_j in $H^0(E(m))$ is $b_j := P_j(m),\ j = 1, \ldots, r$. Let τ represent the isomorphism classes of Γ-modules of dimension $a := P(n)$ and $b := P(m)$ determined by type τ_P.

Fix Γ-modules V and W of dimensions a and b, respectively. We say that a Γ-Kronecker module $\alpha\colon V \otimes H \longrightarrow W$ is of type τ if the pair of Γ-modules (V, W) is of type τ.

Note that the A_Γ-modules of type τ corresponds to the Kronecker-McKay modules of the dimension vector $(\mathbf{a}, \mathbf{b}) := (a_1, \ldots, a_r, b_1, \ldots, b_r)$.

Consider the composition of functors $\Phi := \Psi \circ \Phi_\Gamma\colon \mathbf{Coh}_{\mathbf{\Gamma}}(Y) \longrightarrow \mathbf{Mod}\text{-}B_H$. The next task is to choose the appropriate notion of semistability for Kronecker-McKay modules, so that the functor Φ preserve the notion of semistability.

The preservation of semistability under the functor Φ_Γ is a straightforward generalization of the results of [1, Section 5] (see, [2, Section 3]). The description of the functor Ψ motivates the following:

For a Kronecker-McKay module $N := (V_1, \dots, V_r, W_1, \dots, W_r)$, we define the slope $\mu(N)$ as follows:

$$\mu(N) := \frac{\sum_{i=1}^{r} \dim V_i \cdot \dim \rho_i}{\sum_{j=1}^{r} \dim W_j \cdot \dim \rho_j}\ .$$

Definition 4.9. Let N be a Kronecker-McKay module of dimension vector $(\mathbf{a}, \mathbf{b})$. We say that N is semistable (respectively, stable) if for all non-zero proper B_H-submodules N' of N, we have $\mu(N') \leq \mu(N)$ (respectively, $\mu(N') < \mu(N)$).

For sufficiently large $m \gg n \gg 0$, the following holds [2, Theorem 3.1, Proposition 4.3]

1. A Γ-sheaf $\mathcal{E}$ on Y is semistable if and only if it is pure, n-regular and the Kronecker-McKay module $\Phi(\mathcal{E})$ is semistable.
2. The functor Φ preserves the S-equivalence classes, i.e., the semistable sheave $\mathcal{E}$ and $\mathcal{E}'$ having type τ_P are S-equivalent if and only if the Kronecker-McKay modules $\Phi(\mathcal{E})$ and $\Phi(\mathcal{E}')$ are S-equivalent.

Moduli functors and the representation space

Fix a type τ_P. Consider the moduli functor $\mathcal{M}_Y(\tau_P)\colon \mathbf{Sch}^{\circ} \longrightarrow \mathbf{Set}$ which assigns to each scheme S the set of all isomorphism classes of flat families over S of Γ-sheaves on Y of type τ_P. We denote by $\mathcal{M}_Y^{s}(\tau_P) \subseteq \mathcal{M}_Y^{ss}(\tau_P)$ open sub-functors consists of isomorphism classes of flat families over S of stable (resp. semistable) Γ-sheaves on Y of type τ_P. For sufficiently large enough integer n, the functor $\mathcal{M}_Y^{ss}(\tau_P)$ is an open sub-functor of $\mathcal{M}_Y^{reg}(\tau_P)$ which is defined as a sub-functor of $\mathcal{M}_Y(\tau_P)$ by demanding that all the Γ-sheaves of type τ_P in the flat families are n-regular.

In order to construct the moduli space of semistable Γ-sheaves of type τ_P, we translate the problem into the GIT quotient of certain open subset of a representation space of Kronecker-McKay quiver.

Let

$$\tilde{R} := R_{B_H}(\mathbf{a}, \mathbf{b}) = \bigoplus_{i,j} Hom_k(V_i \otimes H_{ij}, W_j). \tag{4.6}$$

be the representation space of Kronecker-McKay modules of a fixed dimension vector $(\mathbf{a}, \mathbf{b})$. Let $G_\Gamma := \prod_{i,j} GL(V_i) \times GL(W_j)/\Delta$, where as before $\Delta := \{(t\mathbf{1}, \dots, t\mathbf{1}) \,|\, t \in k^{\times}\}$.

Note that the isomorphism classes of Kronecker-McKay modules are in a natural bijection with the orbits of the representation space by the canonical left action of G_Γ, that is, for $g = (g_{0i}, g_{1j})_{i,j}$ and $\alpha = (\alpha_{ij})_{i,j} \in \tilde{R}$,

$$(g \cdot \alpha)_{i,j} := g_{1j} \circ \alpha_{ij} \circ (g_{0i}^{-1} \otimes 1_{H_{ij}}).$$

There are open subsets $\tilde{R}^s \subseteq \tilde{R}^{ss} \subseteq \tilde{R}$ given by the conditions that $\alpha \in \tilde{R}$ is stable or semistable with respect to the semistability notion defined in the Definition 4.9.

Consider the character

$$\chi \colon G_\Gamma \longrightarrow k^*; (g_{0i}, g_{1j}) \mapsto \prod_i \det(g_{0i})^{-b|\rho_i|} \cdot \prod_j \det(g_{1j})^{a|\rho_j|} \tag{4.7}$$

It can be easily checked that the Kronecker-McKay module M is semistable in the sense of Definition 4.9 if and only if the corresponding point $(\alpha_{ij}) \in \tilde{R}$ is χ-semistable (resp. χ-stable). Let $\mathbf{M}^{ss}_{B_H}(\mathbf{a}, \mathbf{b})$ be the good quotient of $\tilde{R}^{ss}$ by G_Γ and $\mathbf{M}^{s}_{B_H}(\mathbf{a}, \mathbf{b})$ be the geometric quotient of $\tilde{R}^s$ by G_Γ Using the general theory of moduli of representations of quivers [18], we have the moduli spaces $\mathbf{M}^{s}_{B_H}(\mathbf{a}, \mathbf{b}) \subseteq \mathbf{M}^{ss}_{B_H}(\mathbf{a}, \mathbf{b})$ of stable and semistable Kronecker-McKay modules of dimension vector $(\mathbf{a}, \mathbf{b})$, respectively (cf. [1, §4.8]).

Construction of moduli of equivariant sheaves

A key point is that the Kronecker-McKay module M is in the image of the functor $\tilde{\Phi}$ is locally closed condition in $\tilde{R}$ ([2, Proposition 2.4, Proposition 5.2]). This enables to prove that the moduli functor $\mathcal{M}^{reg}_Y(\tau_P)$ is locally isomorphic to the quotient functor $\underline{\tilde{Q}}/\underline{G_\Gamma}$, where $\tilde{Q}$ is a locally closed G_Γ-invariant subscheme of $\tilde{R}$ (see, [2, Theorem 5.1]). The open subscheme $\tilde{Q}^{[ss]}$ of $\tilde{Q}$ parametrizing semistable Γ-sheaves of type τ_P is a locally closed subscheme of $\tilde{R}^{ss}$. It turn out that the moduli functor $\mathcal{M}^{ss}_Y(\tau_P)$ is locally isomorphic to the quotient functor $\underline{\tilde{Q}^{[ss]}}/\underline{G_\Gamma}$. Therefore, the problem of construction of moduli scheme $\mathbf{M}^{ss}_Y(\tau_P)$ which corepresent the moduli functor $\mathcal{M}^{ss}_Y(\tau_P)$ reduced to the problem of existence of a good quotient of $\tilde{Q}^{[ss]}$ by G_Γ. The moduli scheme $\mathbf{M}^{ss}_Y(\tau_P)$ is constructed as a good quotient of $\tilde{Q}^{[ss]}$ by G_Γ ([2, Proposition 5.3]) is, a priori, quasi-projective. There is morphism $\phi \colon \mathbf{M}^{ss}_Y(\tau_P) \longrightarrow \mathbf{M}^{ss}_{B_H}(\mathbf{a}, \mathbf{b})$ induced by the inclusion $\tilde{Q}^{[ss]} \subset \tilde{R}^{ss}$. Using the valuative criterion for properness, it is proved that $M^{ss}_Y(\tau_P)$ is proper [2, Proposition 5.4].

As in the non-equivariant case, we have the explicit description of the projective embedding of the moduli space $\mathbf{M}^{ss}_Y(\tau_P)$ into the projective space using certain determinant theta functions. For more detailed account of this, we refer to [2, Section 6]. The main result is as follows:

Theorem 4.10. *[2, Theorem 6.3] For any type τ_P, we can find a Γ-modules U_0, U_1 (satisfying certain condition) and finitely many Γ-equivariant maps*

$$\delta_0, \ldots, \delta_N : U_1 \otimes \mathcal{O}_Y(m) \longrightarrow U_0 \otimes \mathcal{O}_Y(-n)$$

such that the map

$$\Theta_\delta : \mathbf{M}_Y^{ss}(P, \tau_P) \longrightarrow \mathbf{P}^N \ : \ [E] \mapsto (\theta_{\delta_0}(E) : \cdots : \theta_{\delta_N}(E)) \tag{4.8}$$

is a closed scheme-theoretic embedding in characteristic zero, while in characteristic p, it is scheme-theoretic on the stable locus.

4.3. Parabolic bundles

The notion of parabolic structure on vector bundles over a compact Riemann surface was first introduced by C. S. Seshadri and their moduli were constucted by V. B. Mehta and C. S. Seshari [26]. Since then the parabolic bundles and their moduli spaces have been studied quite extensively by several Mathematicians. The notion of parabolic bundles and several other related notions and techniques have been generalized from curves to higher dimensional varieties by Maruyama and Yokogawa [23]. In the following, we will restrict ourselves to describe some results concerning to Faltings parabolic theta functions on curves using the results of §4.2.

Let C be a smooth projective curve defined over the field C of complex numbers. Let $P = x_1 + x_2 + \cdots + x_l$ be a divisor of C. A parabolic bundle E_* on C with parabolic structure over P is a vector bundle E on C equipped with a weighted flag of the fibre over each point $x \in P$. More precisely, a parabolic bundle E_* on C with parabolic structure over P is a vector bundle E on C together with the following data:

(a) for each $x \in P$, a strictly decreasing flag

$$F^\bullet E(x) : E(x) = F^1 E(x) \supset F^2 E(x) \supset \cdots \supset F^{k_x} E(x) \supset F^{k_x+1} E(x) = 0,$$

where $E(x)$ denote the fibre of E over x. The integer k_x is called the length of the flag $F^\bullet E(x)$.

(b) for each $x \in P$, a sequence of real numbers $0 \leq \alpha_1^x < \alpha_2^x < \cdots < \alpha_{k_x}^x < 1$, called the weights associated to the flag $F^\bullet E(x)$.

Let $E_* = (E, F^i E(x), \alpha_i^x)_{x \in P}$ be a parabolic bundle on C. Let

$$r_i^x = \dim F^i E(x) - \dim F^{i+1} E(x)\,.$$

The sequence $(\alpha_1^x, \alpha_2^x, \ldots, \alpha_{k_x}^x)$ is called the type of the flag $F^\bullet E(x)$, and r_i^x is called the multiplicity of the weight α_i^x. By a fixed parabolic type τ_p, we mean a fixed flag type and weights.

We have the notion of parabolic degree defined as follows:

$$pdeg(E) = deg(E) + \sum_{x \in P} \sum_{i=1}^{k_x} r_i^x \alpha_i^x,$$

where $deg(E)$ denotes the topological degree of E.

The parabolic slope, denoted by $p\mu(E_*)$, is defined by

$$p\mu(E_*) = \frac{pdeg(E)}{rank(E)}$$

Thus, we have the notion of semistability for parabolic bundles on C [26]. Recall that any subbundle F of E has the structure of parabolic bundle uniquely determined by that of E. We say that a parabolic vector bundle E_* semistable (respectively, stable) if for every proper subbundle F of the vector bundle E, we have

$$p\mu(F_*) \leq p\mu(E_*) (\text{respectively,} p\mu(F_*) < p\mu(E_*)).$$

For a fixed parabolic type τ_p (with rational weights), let N_0 be a positive integer such that all parabolic weights are integral multiple of $1/N_0$. Then there is an algebraic Galois covering $p\colon \widetilde{C} \longrightarrow C$ with Galois group Γ such that the category of all Γ-equivariant vector bundles over Y is equivalent to the category of all parabolic vector bundles on C with parabolic structure over P with all the weights are integral multiple of $1/N_0$ [26, 8]. For a Γ-equivariant vector E' on $\widetilde{C}$, let $E_* := (p_* E')^\Gamma$ be the corresponding parabolic bundle on C with natural parabolic structure. Then, we have

$$pdeg(E) = \frac{degE'}{N_0}\,. \tag{4.9}$$

Remark 4.11. Let $h\colon Y \longrightarrow C$ be an algebraic Galois covering with Galois group G. Let E be a G-equivariant bundle on Y of rank r, and $y \in Y$. Then E is defined locally at y by a representation of the isotropy group G_y, which is determined uniquely upto isomorphism of representations of G_y.

If $y_1, y_2 \in Y$ such that $h(y_1) = h(y_2)$, then the isotropy groups G_{y_1} and G_{y_2} are conjugate subgroups of G. Hence, a G-equivariant bundle E on Y is defined locally at y_1 and y_2 by a conjugate representations of G_{y_1}. Choose a point $\tilde{x}_i$ of Y over each ramification point x_i in C of a covering h. Let τ represents the isomorphism classes of representations $\varrho_i : G_{\tilde{x}_i} \longrightarrow GL(r, C)$.

We say that a G-equivariant bundle E is locally of type τ, if at each $\tilde{x}_i$, the vector bundle E is locally $G_{\tilde{x}_i}$-isomorphic to the $G_{\tilde{x}_i}$-equivariant bundle defined by ϱ_i.

By fixing a local type τ, the parabolic type τ_p gets automatically fixed and vice-versa, by rigidity.

A cohomological criterion for a parabolic vector bundle on a curve to be semistable is proved in [7]. To state the result precisely, let $V' := \mathcal{O}_{\widetilde{C}} \otimes_k k[\Gamma]$ be the trivial vector bundle on $\widetilde{C}$. Then V' is naturally a Γ-bundle on $\widetilde{C}$. Let V_* be the parabolic vector bundle over C associated to V'.

Theorem 4.12. *[7, Theorem 2.1] A parabolic vector bundle E_* over C is semistable if and only if there is a parabolic vector bundle F_* such that*

$$H^i(C, E_* \otimes F_* \otimes V_*)_0) = 0$$

for all i, where $(E_* \otimes F_* \otimes V_*)_0$ is the underlying vector bundle of the parabolic tensor product $E_* \otimes F_* \otimes V_*$ on C.

Let $\mathcal{E}_*$ be a family of parabolic bundles of parabolic type τ_p on C parametrized by a scheme S. As a consequence of the above criterion, the semistable parabolic bundles in the family $\mathcal{E}_*$ can be detected by the non-vanishing of certain determinant functions, so called Faltings parabolic theta functions. More precisely, if F_* is a parabolic bundle on C, then there is a line bundle $D(\mathcal{E}_*, F_*, V_*)$ on S defined as the determinant line bundle of a 2-term complex $P; \bullet : P^0 \xrightarrow{p} P^1$ of vector bundles on S which computes the cohomology of $(\mathcal{E}_* \otimes F_* \otimes V_*)_0$ locally over S. If $\chi((\mathcal{E}_* \otimes F_* \otimes V_*)_0) = 0$, there is a section θ_{F_*} on S which can be locally identified with $\det p$ over S (cf. §4.1, [3, 5]). For $s \in S$, using Theorem 4.12, it follows that if $\theta_{F_*}(s) \neq 0$, then the parabolic bundle $\mathcal{E}_{*_s}$ is semistable.

As in the usual vector bundle case, the above characterization of semistable parabolic bundles can be used to give a GIT-free construction of the moduli space of semistable parabolic bundles over C. Let us briefly review this construction.

Recall that there is a smooth quasi-projective variety $R(r, d)$ and a family $\mathcal{E}$ of vector bundles of rank r and degree d on C parametrized by $R(r, d)$

such that underlying vector bundles of all semistable parabolic bundles of parabolic type τ_p and degree d occur in this family [26, p. 226] (see §4.1). Let $F(\mathcal{E})$ be the flag variety of type determined by τ_p on $C \times R(r,d)$. Let

$$\widetilde{R} := F(\mathcal{E})|_{P \times R(r,d)}$$

be the restriction of the flag variety $F(\mathcal{E})$ to $P \times R(r,d)$, and denote by $\pi \colon \widetilde{R} \longrightarrow R(r,d)$ the canonical projection. Let $\widetilde{\mathcal{E}}_* := (Id_C \times \pi)^*\mathcal{E}$ be the family of parabolic bundles of parabolic type τ_p on C parametrized by $\widetilde{R}$ obtained by pulling back the family $\mathcal{E}$ of vector bundles of rank r and degree d on C parametrized by $R(r,d)$. Let $\widetilde{R}^{ss}$ be the set of all points $q \in \widetilde{R}$ such that the corresponding parabolic bundle $\widetilde{\mathcal{E}}_{*_q}$ is semistable. It is proved in [5] that it is possible to find finitely many parabolic bundles $F_{0_*}, F_{1_*}, \ldots, F_{N_*}$ which detect all semistable parabolic bundles of parabolic type τ_p and degree d. In other words, the sections $\{\theta_{F_{0_*}}, \theta_{F_{1_*}}, \ldots, \theta_{F_{N_*}}\}$ have no base points in $\widetilde{R}^{ss}$. Hence, they define a morphism

$$\widetilde{\Theta} \colon \widetilde{R}^{ss} \longrightarrow \mathbf{P}^N .$$

Let $M_C^{ss}(\tau_p)$ be the set of S-equivalence classes of semistable parabolic vector bundles of parabolic type τ_p and degree d on C. Then the image $\widehat{M_1} := \widetilde{\Theta}(\widetilde{R}^{ss})$ is a closed in $\mathbf{P}^N$, and hence it is a projective variety (see [5, p. 447]). Let

$$\varphi \colon \widetilde{R}^{ss} \longrightarrow M_C^{ss}(\tau_p)$$

be a set-theoretic map defined by $q \mapsto gr(\mathcal{E}_q)$. Then, the map $\widetilde{\Theta}$ factors through φ [5, Lemma 4.5].

Let $\eta \colon \widehat{M} \longrightarrow \widehat{M_1}$ be the normalization of $\widehat{M_1}$ in the function filed of $\widetilde{R}^{ss}$. There is a set-theoretic bijection $\kappa \colon M_C^{ss}(\tau_p) \longrightarrow \widehat{M}$ [5, p. 447-448]. Hence, we get the structure of normal projective variety on $M_C^{ss}(\tau_p)$. The corresponding morphism on the moduli space

$$\Theta_{F_*} \colon M_C^{ss}(\tau_p) \xrightarrow{\simeq} \widehat{M} \xrightarrow{\eta} \widehat{M_1} \hookrightarrow \mathbf{P}^N$$

is a normalization of the image $\widehat{M_1}$.

Let $M_{\widetilde{C}}^{ss}(\tau)$ be the moduli space of semistable Γ-bundles on $\widetilde{C}$ of local type τ and degree $N_0 d'$, where d' is determined by (4.9) for a fixed parabolic type τ_p (cf. [37], [2]).

By using Seshadri-Biswas correspondence [8], it follows that a set-theoretic map

$$\psi \colon M_{\widetilde{C}}^{ss}(\tau) \longrightarrow M_C^{ss}(\tau_p) \tag{4.10}$$

defined by $[E'] \mapsto [E_*]$, where $E_* = p_*(E')^\Gamma$ with natural parabolic structure, is an isomorphism of schemes.

In [3], using the isomorphism ψ in (4.10), it is proved that the determinant theta functions on $M^{ss}_{\widetilde{C}}(\tau)$ (see Theorem 4.10) coincide with certain Faltings parabolic theta functions which gives an implicit construction of the moduli space $M^{ss}_C(\tau_p)$. Consequently, it follows that the the morphism $\Theta_{F_*}: M^{ss}_C(\tau_p) \longrightarrow \mathbf{P}^N$ is a closed scheme-theoretic embedding (see [3] for more details).

Some questions

It would be interesting to investigate whether one can find an appropriate quiver (which does not depend on the Galois covering $p: \widetilde{C} \longrightarrow C$ and a Galois group Γ) so that its moduli of representations can be used to give a functorial construction of the moduli of semistable parabolic bundles on smooth projective curves or more generally on arbitrary projective varieties.

In view of the Theorem 4.5, Theorem 4.10 and analogous result for parabolic bundles, this functorial approach may be useful in studying the projective normality of such moduli spaces.

The birational classification of moduli spaces of representations of quivers is studied in [31] in which the problem is reduced to the fundamental problem of simultaneous conjugacy of tuples of square matrices. This may be useful in studying the rationality problem of certain moduli spaces of bundles or sheaves.

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