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Generalized centroid of Γ-semirings

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Abstract

We define and study the generalized centroid of a semiprime Γ-semiring. We show that the generalized centroid C_Γ is a multiplicatively reguler Γ-semiring and so Γ-semifield and give some properties of the generalized centroid of a semiprime Γ-semiring.

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1. Introduction

Γ-semirings were first studied by M. Murali Krishna Rao [3] as a generalization of Γ-ring as well as of semiring. All definitions and fundamental concepts concerning Γ-semirings can be found in [3], [4], [1], [2],[5]. and [6]. Öztürk and Jun studied the extended centroid of a prime Γ-ring in [9] and [10]. They also introduced some properties of the generalized centroid of semiprime Γ-ring in [11]. Recently, Yazarlı and Öztürk [7] considered the extended centroid of a prime semiring. After, Öztürk introduced the extended centroid of the prime Γ-semirings [8]. Let S be right multiplicatively cancellable semiprime Γ-semiring. In this paper we consider the main results as follows: (1) The generalized centroid of S is multiplicatively reguler Γ-semiring. (2) C_Γ is a Γ-semifield. (3) $S_\Gamma = S\Gamma C_\Gamma$ is a right multiplicatively cancellable semiprime Γ-semiring. (4) Let S be multiplicatively cancellable semiprime Γ-semiring. If a and b are nonzero elements in S_Γ such that $a\gamma x\beta b = b\beta x\gamma a$ for all $x \in S$ and for all $\gamma, \beta \in \Gamma$ then there exists $q \in C_\Gamma$ and $\gamma \in \Gamma$ such that $q\gamma a = b$. (5) Let $f : S \to S_\Gamma$ be an additive map satisfying $f(x\beta y) = f(x)\beta y$ for all $x, y \in S$ and $\beta \in \Gamma$. Then there exists $q \in Q_r(S_\Gamma)$ such that $f(x) = q\beta x$ for all $x \in S$.

2. Preliminaries

Let S and Γ be two additive commutative semigroups. Then S is called Γ-semiring if there exists a mapping $S \times \Gamma \times S \to S$ (image to be denoted by $a\alpha b$ for $a, b \in S$ and $\alpha \in \Gamma$) satisfying the following conditions for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$:

i) $a\alpha(b + c) = a\alpha b + a\alpha c$
ii) $(a + b)\alpha c = a\alpha c + b\alpha c$
iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$
iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.

A Γ-semiring S is said to have a zero element if there exists an element $0_S \in S$ such that $0_S + x = x = x + 0_S$ and $0_S\gamma x = 0_S = x\gamma 0_S$ for all $x \in S$ and $\gamma \in \Gamma$. Also, a Γ-semiring S is said to be commutative if $x\gamma y = y\gamma x$, for all $x, y \in S$ and $\gamma \in \Gamma$. Let S be a Γ-semiring with zero. If there exists an element $0_\Gamma \in \Gamma$such that $a0_\Gamma b = b0_\Gamma a = 0_S$ for all $a, b \in S$ and $0_\Gamma + \beta = \beta$ for all $\beta \in \Gamma$, then 0_Γ is called the zero of Γ. When the context is clear we simply write 0 instead of 0_Γ. Throughout this paper we consider Γ-semiring with zero.

Let S be a Γ-semiring. An element $a \in S$ is called a left identity (resp.

right identity) of S if $x = a\gamma x$ (resp. $x = x\gamma a$) for all $x \in S$ and $\gamma \in \Gamma$. If a is both a left and right identity, then a is called an identity of S. In this case we say that S is a Γ-semiring with identity . If A and B are subsets of a Γ-semiring S and $\Delta \subset \Gamma$, we denote by $A\Delta B$, the subset of S consisting of all finite sums of the form $\sum a_i\alpha_i b_i$ where $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Delta$. For the singleton subset $\{x\}$ of S we write $x\Delta A$ instead of $\{x\}\Delta A$.

A nonempty subset I of a Γ-semiring S is called a subΓ-semiring of S if I is a subsemigroup of $(S,+)$ and $a\gamma b \in I$ for all $a, b \in I$ and $\gamma \in \Gamma$. A right (left) ideal I of a Γ-semiring S is an additive subsemigroup of S such that $I\,\Gamma S \subset I$ ($S\Gamma I \subset I$). If I is both a right and a left ideal of S, then we say that I is a two-sided ideal or simply an ideal of S.

Let S be a Γ- semiring. A proper ideal P of S is said to be semiprime if for any ideal A of S, $A\Gamma A \subseteq P$ implies that $A \subseteq P$. A Γ-semiring S is called a semiprime Γ-semiring if $< 0_S >$ is a semiprime ideal of S.

Theorem 1. [1, Theorem 3.6.] If P is an ideal of a Γ-semiring S then the following conditions are equivalent:

i) P is semiprime.
ii) If $a \in S$ such that $a\Gamma S\Gamma a \subseteq P$ then $a \in P$.
iii) For $a \in S$ if $< a > \Gamma < a > \subseteq P$ then $a \in P$.
iv) If U is a right ideal of S such that $U\Gamma U \subseteq P$ then $U \subseteq P$.
v) If V is a left ideal of S such that $V\Gamma V \subseteq P$ then $V \subseteq P$.

A commutative Γ-semiring S is said to be Γ-semifield if for any $a(\neq 0) \in S$ and for any $\alpha \in \Gamma$ there exists $b \in S$, $\beta \in \Gamma$ such that $a\alpha b\beta d = d$ for all $d \in S$.

An element a of Γ-semiring S is regular if there exists $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$. A Γ-semiring S is regular if every element in S is regular.

Let S be a Γ-semiring. A commutative monoid $(M,+)$ with additive identity 0_M is said to be a right Γ-semiring S-semimodule or simply a ΓS-semimodule, if there exists a mapping $M \times \Gamma \times S \to M$ (images to be denoted by $a\alpha S$ for $a \in M, \alpha \in \Gamma$, $s \in S$) satisfying the following conditions for all $a, b \in M$, for all $s, t \in S$ and for all $\alpha, \beta \in \Gamma$:

i) $(a + b)\alpha s = a\alpha s + b\alpha s$,
ii) $a\alpha(s + t) = a\alpha s + a\alpha t$,
iii)$a(\alpha + \beta)s = a\alpha s + a\beta s$,
iv) $a\alpha(s\beta t) = (a\alpha s)\beta t$,
v) $0_M\alpha s = 0_M = a\alpha 0_S$.

One defines a left Γ-semiring S-semimodule in an analogous fashion. Let R and S both be Γ-semirings and f a map of R into S. Then f is a Γ-homomorphism if and only if $f(r_1+r_2) = f(r_1) + f(r_2)$ and $f(r_1\gamma r_2) = f(r_1)\gamma f(r_2)$ for all $r_1,r_2 \in R$ and for all $\gamma \in \Gamma$. A Γ-homomorphism of semirings which is both injective and surjective is called isomorphism. If there exists isomorphism between Γ-semirings R and S we write $R \cong S$. If $f : R \to S$ is Γ-homomorphism of semirings, then $Im(f) = \{ f(r) | r \in R\}$ is Γ-subsemiring of S.

Let S be Γ-semiring, M and N be ΓS-semimodule. Then a function f from M to N is a right ΓS-semimodule homomorphism if and only if the following conditions are satisfied:

i) $f(m_1 + m_2) = f(m_1) + f(m_2)$ for all $m_1,m_2 \in M$,

ii) $f(m\alpha s) = f(m)\alpha s$ for all $m \in M$, for all $s \in S$ and for all $\alpha \in \Gamma$.

3. Generalized centroid

Definition 1. *Let S be Γ-semiring. For a subset U of S,*

$$Ann_l U = \{ a \in S | a\Gamma U = \langle 0_S \rangle \}$$

is called the left annihilator of U. A right annihilator $Ann_r U$ can be defined similarly.

Lemma 1. *Let S be a semiprime Γ-semiring and U a non-zero ideal of S. Then $Ann_l U = Ann_r U$ and in this case we will write $Ann_l U = Ann_r U = AnnU$. Also $AnnU \cap U = \langle 0_S \rangle$.*

Proof. It is clear that $Ann_l U$ and $Ann_r U$ are ideals of S. Since $Ann_l U \Gamma U = \langle 0_S \rangle$, $U\Gamma Ann_l U \Gamma U \Gamma Ann_l U = \langle 0_S \rangle$. Since S is semiprime Γ-semiring, we get $U\Gamma Ann_l U = \langle 0_S \rangle$. That is, $Ann_l U \subseteq Ann_r U$. On the other hand, since $U\Gamma Ann_r U = \langle 0_S \rangle$, we have $Ann_r U \Gamma U \Gamma Ann_r U \Gamma U = \langle 0_S \rangle$. Then, $Ann_r U \Gamma U = \langle 0_S \rangle$. Hence $Ann_r U \subseteq Ann_l U$, and so, $Ann_l U = Ann_r U$.

Since $AnnU \cap U$ is an ideal of S and $(AnnU \cap U)\Gamma(AnnU \cap U) \subseteq U\Gamma AnnU = \langle 0_S \rangle$, we have $AnnU \cap U = \langle 0_S \rangle$, since S is semiprime Γ-semiring. □

Lemma 2. *Let S be a semiprime Γ-semiring. Let us denote by F a set of all ideals of S which have zero annihilator in S. In this case, the set F is closed under multiplication.*

Proof. Let U and V be in F. The equality $U\Gamma V\beta x = \langle 0_S\rangle$ for $x \in S$ and all $\beta \in \Gamma$ yields $V\beta x \subseteq Ann_r U = \langle 0_S\rangle$, i. e., $V\beta x = \langle 0_S\rangle$ and so $x \in Ann_r V = \langle 0_S\rangle$ which implies $x = 0_S$. Then we get $U\Gamma V \in F$. □

Lemma 3. *Let S be a semiprime Γ-semiring and U a nonzero ideal of S. Then $U \in F$ if and only if U has nonzero intersection with any nonzero ideal of S.*

Proof. Let $U \in F$. Then $\langle 0_S\rangle \neq U\Gamma V \subseteq U \cap V$ where V is nonzero ideal of S.

Conversely, since $U \cap AnnU = \langle 0_S\rangle$, then $AnnU = \langle 0_S\rangle$ and so $U \in F$. □

Remark 1. *If U, $V \in F$, then $U \cap V \in F$.*

Let S be a semiprime Γ-semiring such that $S\Gamma S \neq S$. The ideals U of S and S regard as right ΓS-semimodules. Denote

$M \;:= \{\, f : U \to S \mid <0_S> \neq U$ is ideal of S,
f is right ΓS-semimodule homomorphism $\}$.

Define a relation $\sim$ on M by $f \sim g \Leftrightarrow \exists\ K(\in F) \subseteq U \cap V$ such that $f = g$ on K where $U$and V are domains of f and g respectively. Since the set F is closed under multiplication, it is possible to find a nonzero K and so " $\sim$ " is an equivalence relation.

This gives a chance for us to get a partition of M. Then we denote the equivalence class by $\widehat{f} = [U, f]$, where $\widehat{f} := \{g : V \to S \mid f \sim g\}$ and denote by Q_r set of all equivalence classes. That is,

$Q_r \;= \{\widehat{f} \mid f : U \to S$ is right ΓS-semimodule homomorphism and
$<0_S> \neq U$ is ideal of S $\}$.

Now we define an addition " $+$ " on Q_r as follows:

$$\widehat{f} + \widehat{g} = \widehat{f + g}$$

for all $\widehat{f}, \widehat{g} \in Q_r$. Let $\widehat{f}, \widehat{g} \in Q_r$ where U and V are domains of f and g respectively. Therefore $f + g : U \cap V \to S$ is a right ΓS-semimodule homomorphism. Assume that $f_1 \sim f_2$ and $g_1 \sim g_2$ where U_1, U_2, V_1 and V_2 are domains of f_1, f_2, g_1 and g_2 respectively. Then $\exists K_1(\in F) \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1 and $\exists K_2(\in F) \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2. Taking $K = K_1 \cap K_2$. Then $K \in F$ and

$$\begin{aligned} K &= K_1 \cap K_2 \subseteq (U_1 \cap U_2) \cap (V_1 \cap V_2) \\ &= (U_1 \cap V_1) \cap (U_2 \cap V_2) . \end{aligned}$$

For any $x \in K$, we have $(f_1 + g_1)(x) = f_1(x) + g_1(x) = f_2(x) + g_2(x) = (f_2 + g_2)(x)$, and so $f_1 + g_1 = f_2 + g_2$ on K. Therefore $f_1 + g_1 \sim f_2 + g_2$ where $f_1 + g_1 : U_1 \cap V_1 \to S$ and $f_2 + g_2 : U_2 \cap V_2 \to S$ are right ΓS-semimodule homomorphisms. That is, addition "+" is well-defined. Now we prove that Q_r is a commutative monoid. It is shown easly that $\widehat{f} + \left(\widehat{g} + \widehat{h}\right) = \left(\widehat{f} + \widehat{g}\right) + \widehat{h}$ and $\widehat{f} + \widehat{g} = \widehat{g} + \widehat{f}$ for all $\widehat{f}, \widehat{g}, \widehat{h} \in Q_r$.

Taking $\widehat{\theta} \in Q_r$ where $\theta : S \to S$, $x \mapsto 0_S$ for all $x \in S$. Let $\widehat{f} \in Q_r$, where U is domain of f. Since $U \subseteq U \cap S$, we get for all $x \in U$,

$$(f + \theta)(x) = f(x) + \theta(x) = f(x) + 0_S = f(x)$$

and

$$(\theta + f)(x) = \theta(x) + f(x) = 0_S + f(x) = f(x) .$$

Thus, $\widehat{f} + \widehat{\theta} = \widehat{\theta} + \widehat{f} = \widehat{f}$. Hence $\widehat{\theta}$ is the additive identity in Q_r.

Since $S\Gamma S \neq S$ and S is a semiprime Γ-semiring, $S\Gamma S(\neq < 0_S >)$ is an ideal of S. Therefore $S\beta S \in F$ for every $\beta(\neq 0) \in \Gamma$. We can take the homomorphism $1_{S\Gamma} : S\Gamma S \to S$ as a unit ΓS-semimodule homomorphism. Note that $S\beta S \neq < 0 >$ for all $< 0 > \neq \beta \in \Gamma$ so that $1_{S\beta} : S\beta S \to S$ is nonzero ΓS-semimodule homomorphism. Denote

$$N := \{ 1_{S\beta} : S\beta S \to S \mid 0 \neq \beta \in \Gamma \},$$

and define a relation " $\approx$ " on N by $1_{S\beta} \approx 1_{S\gamma} \Leftrightarrow \exists\, W := S\alpha S(\neq < 0 >) \subseteq S\beta S \cap S\gamma S$ such that $1_{S\beta} = 1_{S\gamma}$ on W where $S\beta S$ and $S\gamma S$ are domains of $1_{S\beta}$ and $1_{S\gamma}$ respectively. We can easily check that " $\approx$ " is an equivalence relation on N. Denote by $\widehat{\beta} = [S\beta S, 1_{S\beta}]$, the equivalence class containing $1_{S\beta}$ and by $\widehat{\Gamma}$ the set of all equivalence classes of N with respect to " $\approx$ ", that is,

$$\widehat{\beta} := \{ 1_{S\gamma} : S\gamma S \to S \mid 1_{S\beta} \approx 1_{S\gamma} \}$$

and $\widehat{\Gamma} := \{\widehat{\beta} \mid 0 \neq \beta \in \Gamma\}$. Define an addition $+$ on $\widehat{\Gamma}$ as follows:

$$\widehat{\beta} + \widehat{\gamma} = \widehat{\beta + \gamma}$$

for all $\beta(\neq 0), \gamma(\neq 0) \in \Gamma$. Then it is routine to check that $(\widehat{\Gamma}, +)$ is commutative monoid.

Now we define a map $(-,-,-) : Q_r \times \widehat{\Gamma} \times Q_r \to Q_r, (\widehat{f},\ \widehat{\beta}\ , \widehat{g}\ ) \mapsto \widehat{f}\ \widehat{\beta}\ \widehat{g}$, as follows:

$$\widehat{f}\ \widehat{\beta}\ \widehat{g} = \widehat{f\beta g}$$

where U, V and $S\beta S$ are domains of f , g and $1_{S\beta}$ respectively. Therefore $f1_{S\beta}g : V\Gamma S\beta S\Gamma U \to S$ is a right ΓS-semimodule homomorphism where

$$V\Gamma S\beta S\Gamma U = \{ \sum_{finite} v_i\gamma_i s_i\beta r_i\alpha_i u_i \ \mid v_i \in V, u_i \in U, s_i, r_i \in S \text{ and } \gamma_i, \alpha_i\Gamma \ \},$$

an ideal of S. Assume that $f_1 \sim f_2$, $g_1 \sim g_2$ and $1_{S\beta} \approx 1_{S\beta'}$ where $U_1, U_2, V_1, V_2, S\beta S$ and $S\beta' S$ are domains of $f_1, f_2, g_1,\ g_2, 1_{S\beta}$ and $1_{S\beta'}$ respectively. Then $\exists K_1 \subseteq U_1 \cap U_2$ such that $f_1 = f_2$ on K_1, $\exists K_2 \subseteq V_1 \cap V_2$ such that $g_1 = g_2$ on K_2 and $\exists W \subseteq S\beta S \cap S\beta' S$ such that $1_{S\beta} = 1_{S\beta'}$ on W. Also $V_1\Gamma S\beta S\Gamma U_1 \cap V_2\Gamma S\beta' S\Gamma U_2 \subseteq (U_1 \cap S\beta S \cap V_1) \cap \left(U_2 \cap S\beta' S \cap V_2\right) = (U_1 \cap U_2) \cap (V_1 \cap V_2) \cap (S\beta S \cap S\beta' S)$ and there exists $< 0_S > \neq K$ is an ideal of S such that $K \subseteq V_1\Gamma S\beta S\Gamma U_1 \cap V_2\Gamma S\beta' S\Gamma U_2$. For any $x \in K$, $x \in V_1\Gamma S\beta S\Gamma U_1 \cap V_2\Gamma S\beta' S\Gamma U_2$. Hence $x \in V_1\Gamma S\beta S\Gamma U_1$ and $x \in V_2\Gamma S\beta' S\Gamma U_2$. Then, $x = \sum_{finite} v_i\gamma_i s_i\beta r_i\alpha_i u_i$; $v_i \in V_1 \cap V_2$, $u_i \in U_1 \cap U_2$, $s_i, r_i \in S$ and $\gamma_i, \alpha_i\Gamma$. Therefore

$$\begin{aligned}
(f_1 1_{S\beta} g_1)(x) &= f_1(1_{S\beta}(g_1(\sum_{finite} v_i\gamma_i s_i\beta r_i\alpha_i u_i))) \\
&= f_1(g_1(\sum_{finite} v_i\gamma_i s_i\beta r_i\alpha_i u_i)) \\
&= f_1(\sum_{finite} g_1(v_i)\gamma_i s_i\beta r_i\alpha_i u_i) = f_1(\sum_{finite} g_2(v_i)\gamma_i s_i\beta r_i\alpha_i u_i) \\
&= f_2(\sum_{finite} g_2(v_i)\gamma_i s_i\beta r_i\alpha_i u_i) = f_2(g_2(\sum_{finite} v_i\gamma_i s_i\beta r_i\alpha_i u_i)) \\
&= f_2(1_{S\beta'}(g_2(\sum_{finite} v_i\gamma_i s_i\beta r_i\alpha_i u_i))) = \left(f_2 1_{S\beta'} g_2\right)(x)
\end{aligned}$$

and so $f_1 1_{S\beta} g_1 = f_2 1_{S\beta'} g_2$ on K. Hence, $\widehat{f_1}\ \widehat{\beta}\ \widehat{g_1} = \widehat{f_2}\ \widehat{\beta}'\ \widehat{g_2}$. That is, "." is well-defined. Now we will prove that Q_r is a $\widehat{\Gamma}$-semiring with identity.

Let $\widehat{f}, \widehat{g}, \widehat{h} \in Q_r$ where U, V and W are domains of f , g and h respectively and $\widehat{\gamma} \in \widehat{\Gamma}$ where $S\gamma S$ is domains of $1_{S\gamma}$. Since $(V \cap W)\Gamma S\gamma S\Gamma U \subseteq V\Gamma S\gamma S\Gamma U \cap W\Gamma S\gamma S\Gamma U$, we get for all $x \in (V \cap W)\Gamma S\gamma S\Gamma U$,

$$\begin{aligned}[f1_{S\gamma}(g+h)](x) &= f(1_{S\gamma}(g+h)(x)) \\ &= f(1_{S\gamma}(g(x)+h(x))) \\ &= f(1_{S\gamma}(g(x))+1_{S\gamma}(h(x))) \\ &= f(1_{S\gamma}(g(x)))+f(1_{S\gamma}(h(x))) \\ &= (f1_{S\gamma}g)(x)+(f1_{S\gamma}h)(x) \\ &= [f1_{S\gamma}g+f1_{S\gamma}h](x).\end{aligned}$$

Hence $f1_{S\gamma}(g+h) = f1_{S\gamma}g+f1_{S\gamma}h$ on $(V)\,\Gamma S\gamma S\Gamma U$.That is, $\widehat{f}\widehat{\gamma}\left(\widehat{g}+\widehat{h}\right) = \widehat{f}\ \widehat{\gamma}\widehat{g}+\widehat{f}\ \widehat{\gamma}\widehat{h}$. Similarly, the equalities $(\widehat{f}+\widehat{g})\widehat{\gamma}\ \widehat{h} = \widehat{f}\widehat{\gamma}\ \widehat{h}+\widehat{g}\widehat{\gamma}\ \widehat{h}$ and $\widehat{f}(\ \widehat{\gamma}+\widehat{\beta})\ \widehat{g} = \widehat{f}\widehat{\gamma}\ \widehat{g}+\widehat{f}\widehat{\beta}\ \widehat{g}$ are proved in analogous way. Also, let $\widehat{f},\widehat{g},\widehat{h} \in Q_r$ where U, V and W are domains of f , g and h respectively and $\widehat{\gamma},\widehat{\beta} \in \widehat{\Gamma}$ where $S\gamma S, S\beta S$ are domains of $1_{S\gamma}, 1_{S\beta}$ respectively. Since $W\Gamma S\beta S\Gamma(V\Gamma S\gamma S\Gamma U) = (W\Gamma S\beta S\Gamma V)\Gamma S\gamma S\Gamma U$, we get for all $x \in W\Gamma S\beta S\Gamma(V\Gamma S\gamma S\Gamma U)$,

$$\begin{aligned}[(f\ 1_{S\gamma}g)1_{S\beta}h](x) &= ((f1_{S\gamma}g)1_{S\beta})(h(x)) \\ &= f(1_{S\gamma}g(1_{S\beta}h(x))) \\ &= f(1_{S\gamma}(g1_{S\beta}h)(x)) \\ &= [f1_{S\gamma}(g1_{S\beta}h)](x).\end{aligned}$$

Hence $(f\ 1_{S\gamma}g)1_{S\beta}h = f1_{S\gamma}(g1_{S\beta}h)$ on $W\Gamma S\beta S\Gamma(V\Gamma S\gamma S\Gamma U)$. That is, $(\widehat{f}\ \widehat{\gamma}\widehat{g})\widehat{\beta}\widehat{h} = \widehat{f}\ \widehat{\gamma}(\widehat{g}\widehat{\beta}\widehat{h})$. Next we will show that Q_r has an identity. Taking $\widehat{I} \in Q_r$ where $I : S \to S$, $s \mapsto s$ for all $s \in S$. Let $\widehat{f} \in Q_r$, where U is domain of f and $\widehat{\gamma} \in \widehat{\Gamma}$ where $S\gamma S$ is domains of $1_{S\gamma}$ Since $S\Gamma U \subseteq U$, we get for all $x \in S\Gamma S\gamma S\Gamma U$, $(f1_{S\gamma}\ I)(x) = f(1_{S\gamma}(\ I(x))) = f(x)$ and $(I1_{S\gamma}f)(x) = I(1_{S\gamma}(f(x))) = f(x)$. Thus, $\widehat{f}\widehat{\gamma}\widehat{I} = \widehat{I}\widehat{\gamma}\widehat{f} = \widehat{f}$. Hence $\widehat{I}$ is the multiplicative identity in Q_r. Therefore $(Q_r,+,.)$ is a $\widehat{\Gamma}$-semiring with identity. Moreover we have that $\widehat{\theta} \neq \widehat{I}$.

Finally, noticing that the mapping $\phi : \Gamma \to \widehat{\Gamma}$ defined by $\phi(\gamma) = \widehat{\gamma}$ for every $0 \neq \gamma \in \Gamma$ is an isomorphism, we know that the $\widehat{\Gamma}$-ring Q_r is a Γ-semiring. Thus, $(Q_r,+,.)$ be a Γ-semiring. One can, of course, characterize Q_l, the left quotient Γ-semiring of S in a similar manner.

Definition 2. *A Γ-semiring S is said to be right (left) multiplicatively cancellable if $x\gamma y = z\gamma y$; (resp. $x\gamma y = x\gamma z$) for all $x, y, z \in S$ and for all $\gamma \in \Gamma$ implies that $x = z$ (resp. $y = z$).*

Let S is a semiprime Γ-semiring. If S is right multiplicatively cancellable semiring, then S may be embedded in Q_r as a subΓ-semiring. Let $a \in S$. Define $\lambda_{a\gamma} : S \to S$ by $\lambda_{a\gamma}(s) = a\gamma s$ for all $s \in S$ and for all $\gamma \in \Gamma$. It is clear that $\lambda_{a\gamma}$ is a right ΓS-semimodule homomorphism, so that $\lambda_{a\gamma}$ defines element $\widehat{\lambda}_{a\gamma}$ of Q_r. Hence we may define $\psi : S \rightarrow Q_r$ by $\psi(a) = \widehat{\lambda}_{a\gamma}$ for $a \in S$. ψ is a monomorphism.

Therefore S is subΓ-semiring of Q_r. We call Q_r the right quotient Γ-semiring of S. For purposes of convenience, we use q instead of $\hat{q} \in Q_r$.

Definition 3. *The set*

$$C_\Gamma := \{q \in Q_r |\, q\gamma p = p\gamma q \text{ for all } p \in Q_r \text{ and for all } \gamma \in \Gamma\}$$

is called the generalized centroid of a Γ*-semiring* S.

Remark 2. *Assume that* $q = [U, f] \in C_\Gamma$. *For all* $s \in S$, $[S, \lambda_s] . [S\beta S, 1_{S\beta}] . [U, f] = [U, f] . [S\beta S, 1_{S\beta}] . [S, \lambda_s]$ *and so there exists* $K (\in F) \subseteq U\Gamma S\beta S\Gamma S \cap S\Gamma S\beta S\Gamma U$ *such that* $\lambda_s 1_{S\beta} f = f 1_{S\beta} \lambda_s$ *on* K. *From here,* $(\lambda_s 1_{S\beta} f)(x) = (f 1_{S\beta} \lambda_s)(x)$ *for all* $x \in K$, *i.e.,* $s\beta f(x) = f(s\beta x)$. *Hence* f *acts as a* ΓS*-semimodule homomorphism on* K.

The following theorem characterizes the quotient Γ-semiring Q_r of S. The proof is same the proof of the corresponding theorem in ring theory and we omit it.

Theorem 2. Let S be a right multiplicatively cancellable semiprime Γ-semiring and Q_r the quotient Γ-semiring of S. Then the Γ-semiring Q_r satisfies the following properties:
(i) Q_r is semiprime Γ-semiring.
(ii) For any element q of Q_r, there exists an ideal of $U_q \in F$ which has zero annihilator with a right ΓS-semimodule homomorphism $q : U \rightarrow S$, such that $q(U_q) \subseteq S$ (or $q\gamma U_q \subseteq S$ for all $\gamma \in \Gamma$).
(iii) If $q \in Q_r$ and $q(U_q) = \{0_S\}$ for a certain $U_q \in F$ ($q\gamma U_q = \{0_S\}$ for a certain $U_q \in F$ and for all $\gamma \in \Gamma$), then $q = 0$.
(iv) If $U \in F$ and $\Psi : U \rightarrow S$ is a right ΓS-semimodule homomorphism, then there exists an element $q \in Q_r$ such that $\Psi(u) = q(u)$ for all $u \in U$ (or $\Psi(u) = q\gamma u$ for all $u \in U$ and for all $\gamma \in \Gamma$).
(v) Let W be a subΓS-semimodule (an (S, S) subbiΓS-semimodule) in Q_r and $\Psi : W \rightarrow Q_r$ a right ΓS-semimodule homomorphism. If W contains the ideal U of S such that $\Psi(U) \subseteq S$ and $AnnU = Ann_r W$, then there is an element $q \in Q_r$ such that $\Psi(b) = q(b)$ for any $b \in W$ (or $\Psi(b) = q\gamma b$ for any $b \in W$ and $\gamma \in \Gamma$) and $q(a) = 0$ for any $a \in Ann_r W$ (or $q\gamma a = 0$ for any $a \in Ann_r W$ and $\gamma \in \Gamma$).

Theorem 3. Let S be a right multiplicatively cancellable semiprime Γ-semiring and C_Γ the generalized centroid of S. Then all elements of C_Γ are multiplicatively regular.

Proof. Let a be an element of C_Γ. Then $a, a^2 \in Q_r$ and so we get that U_a and U_{a^2} are nonzero ideals which have zero annihilators in S. Hence $J = U_a \cap U_{a^2} \in F$ we consider the mapping $\Psi : J \to S$ defined by $\Psi(a^2\beta x) = a\beta x$ for any $\beta \in \Gamma$ where x runs through the set J. Let $a^2\beta x = a^2\beta y$. Since $a^2 \in C_\Gamma$, $x\beta a^2 = y\beta a^2$. Let $a^2 = [U_{a^2}, f]$ and so $[S, \lambda_x] . [S\beta S, 1_{S\beta}] . [U_{a^2}, f] = [S, \lambda_y] . [S\beta S, 1_{S\beta}] . [U_{a^2}, f]$. Therefore there exists $K \in F$ such that $K \subseteq U_{a^2}\Gamma S\beta S\Gamma S$ and $\lambda_x 1_{S\beta} f = \lambda_y 1_{S\beta} f$ on K. For all $z \in K$, $(\lambda_x 1_{S\beta} f)(z) = (\lambda_y 1_{S\beta} f)(z)$ and so $x\beta f(z) = y\beta f(z)$. Since S be right multiplicatively cancellable Γ-semiring, we get $x = y$. Thus $a\beta x = a\beta y$. That is, Ψ is well-defined. It is easy to see that Ψ is right ΓS-semimodule homomorphism. There exists $a_1 \in Q_r$ such that $a_1\alpha a^2\beta x = a\beta x$ for all $x \in J$. We have that $a_1\alpha a^2 = a$. Let us prove that the element a_1 in C_Γ. Let q be an arbitrary element of Q_r. Then $(a_1\alpha a^2)^2\beta q = q\beta(a_1\alpha a^2)^2$ and so $a^4\alpha a_1^2\beta q = a^4\alpha q\beta a_1^2$. Multiplying this equality from left by $a\alpha a_1^3$, we get $a\alpha a_1\beta q = a\alpha q\beta a_1$. Assume that $a = [U_a, d]$, $a_1\beta q = [V, g]$ and $q\beta a_1 = [H, h]$. Therefore $[V, g] . [S\alpha S, 1_{S\alpha}] . [U_a, d] = [H, h] . [S\alpha S, 1_{S\alpha}] . [U_a, d]$ and so there exists $L(\in F) \subseteq U_a\Gamma S\alpha S\Gamma V \cap U_a\Gamma S\alpha S\Gamma H$ such that $g1_{S\alpha}d = h1_{S\alpha}d$ on L. Since $a \in C_\Gamma$, there exists $W \in F$ such that d is a ΓS-semimodule homomorphism on W. On the other hand $d^{-1}(W \cap L)$ is an ideal which has zero annihilator in S, of S. Also $gd = hd$ on $d^{-1}(W \cap L) \cap L$. Hence $g = h$ on $W \cap L$. That is, $a_1\beta q = q\beta a_1$. This completes the proof. □

Lemma 4. *C_Γ is multiplicatively cancellable Γ-semiring.*

Proof. Let $s\beta p = s\beta q$ for $p, q, s \in C_\Gamma$. Then

$$[H, h] . [S\beta S, 1_{S\beta}] . [U, f] = [H, h] . [S\beta S, 1_{S\beta}] . [V, g]$$

where $p = [U, f]$, $q = [V, g]$, $s = [H, h]$. Hence there exists $(\{0_S\} \neq)K \in F$ such that $K \subseteq U\Gamma S\beta S\Gamma H \cap V\Gamma S\beta S\Gamma H$ and $h1_{S\beta}f = h1_{S\beta}g$ on K. Since $f, g, h \in C_\Gamma$, there exists $W \in F$ such that $f(x)\beta h(y) = g(x)\beta h(y)$ for all $x, y \in K \cap W$. And so $f = g$ on $K \cap W$. That is, $p = q$. Thus C_Γ is multiplicatively cancellable Γ-semiring. □

We have showed that all elements of C_Γ are multiplicatively regular. For any element $a \in C_\Gamma$, there exists an element a_1 in C_Γ such that $a_1 \beta a^2 = a$. Since C_Γ is multiplicatively cancellable Γ-semiring, $a_1 \beta a = I$. Thus all nonzero elements of C_Γ have multiplicative inverse. Thus we have the following result:

Corollary 1. *C_Γ is a Γ-semifield.*

We now let $S_\Gamma = S\Gamma C_\Gamma$, a subΓ-semiring of Q_r containing S. We shall call S_Γ the central closure of S. The same proof used in showing that Q_r was semiprime may be employed to show that S_Γ is semiprime.

Proposition 1. *Let S be right multuplicatively cancellable semiprime Γ-semiring and S_Γ be the central closure of S. Then S_Γ is a right multiplicatively cancellable semiprime Γ-semiring.*

Proof. The proof is similar with the proof of [7, Proposition-2]. But we notice that ideals are in F in the proof. □

Theorem 4. Let S be multiplicatively cancellable semiprime Γ-semiring. If a and b are nonzero elements in S_Γ such that $a\gamma x\beta b = b\beta x\gamma a$ for all $x \in S$ and for all $\gamma, \beta \in \Gamma$ then there exists $q \in C_\Gamma$ and $\gamma \in \Gamma$ such that $q\gamma a = b$.

Proof. The proof is similar with the proof of [7, Theorem-3]. But we notice that ideals are in F in the proof. □

Theorem 5. Let $f : S \to S_\Gamma$ be an additive map satisfying $f(x\beta y) = f(x)\beta y$ for all $x, y \in S$ and $\beta \in \Gamma$. Then there exists $q \in Q_r(S_\Gamma)$ such that $f(x) = q\beta x$ for all $x \in S$.

Proof. Let us extend f from S to S_Γ according to $\overline{f}\left(\sum x_i\alpha_i\lambda_i\right) = \sum f(x_i)\alpha_i\lambda_i$, where $x_i \in S$, $\alpha_i \in \Gamma$ and $\lambda_i \in C_\Gamma$. Let $\sum x_i\alpha_i\lambda_i = \sum y_i\alpha_i\beta_i$, $x_i, y_i \in S, \lambda_i, \beta_i \in C_\Gamma$. There exists a nonzero ideal K in S such that $\lambda_i\alpha K \subseteq S$ for every i. For $a \in K$, the sum $\sum x_i\alpha_i(\lambda_i\gamma a)$ in S. Then,

$\sum x_i\alpha_i(\lambda_i\gamma a) = \sum y_i\alpha_i(\beta_i\gamma a)$

$\sum f(x_i)\alpha_i\lambda_i\gamma a = \sum f(y_i)\alpha_i\beta_i\gamma a$

$\left(\sum f(x_i)\alpha_i\lambda_i\right)\gamma a = \left(\sum f(y_i)\alpha_i\beta_i\right)\gamma a.$

Since S_Γ is a right multiplicatively cancellable semiring, we get $\sum f(x_i)\alpha_i\lambda_i = \sum f(y_i)\alpha_i\beta_i$. And so, $\overline{f}\,(\sum x_i\alpha_i\lambda_i) = \overline{f}\,(\sum y_i\alpha_i\beta_i)$. That is, $\overline{f}$ is well-defined. The fact that $\overline{f}(x\alpha y) = \overline{f}(x)\alpha y$ for all $x, y \in S_\Gamma$ can be seen by a direct computation. Thus $\overline{f} : S_\Gamma \rightarrow S_\Gamma$ is a right S_Γ homomorphism, hence there exists $q \in Q_r(S_\Gamma)$ such that $\overline{f}(x) = q\beta x$, $x \in S$. Since $\overline{f}$ is an extension of f, this proves the theorem. □

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