Further results on 3-product cordial labeling

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Abstract

A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of vertices labeled with i , $e_f(i)$ denotes the number of edges xy with $f(x)f(y) \equiv i \pmod{3}$. A graph with 3-product cordial labeling is called 3-product cordial graph. In this paper we establish that switching of an apex vertex in closed helm, double fan, book graph $K_{1,n} \times K_2$ and permutation graph $P(K_2 + mK_1, I)$ are 3-product cordial graphs.

Key Words. cordial labeling, product cordial labeling, 3-product cordial labeling, 3-product cordial graph.

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1. Introduction

All graphs considered here are simple, finite, connected and undirected. For basic notations and terminology, we follow [3]. A graph labeling is an assignment of integers to the vertices or edges or both, subject to certain conditions. There are several types of labeling and a complete survey of graph labeling is available in [2]. Cordial labeling is a weaker version of graceful labeling and harmonious labeling introduced by Cahit in [1]. Let f be a function from the vertices of G to $\{0, 1\}$ and for each edge xy assign the label $|f(x) - f(y)|$. f is called a cordial labeling of G if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1, and the number of edges labeled 0 and the number of edges labeled 1 differ at most by 1. Sundaram et al. introduced the concept of product cordial labeling in [10]. Let f be a function from $V(G)$ to $\{0, 1\}$. For each edge uv , assign the label $f(u)f(v)$. Then f is called product cordial labeling if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$ where $v_f(i)$ and $e_f(i)$ denotes the number of vertices and edges respectively labeled with i ($i = 0, 1$). The same authors have introduced the concept of EP -cordial labeling in [11]. A vertex labeling $f : V(G) \rightarrow \{-1, 0, 1\}$ is said to be an EP -cordial labeling if it induces the edge labeling f^* defined by $f^*(uv) = f(u)f(v)$ for each $uv \in E(G)$ and if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i \neq j$, $i, j \in \{-1, 0, 1\}$, where $v_f(x)$ and $e_f(x)$ denotes the number of vertices and edges of G having the label $x \in \{-1, 0, 1\}$. In [11] it is remarked that any EP -cordial labeling is a 3-product cordial labeling. A mapping $f : V(G) \rightarrow \{0, 1, 2\}$ is called 3-product cordial labeling if $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for any $i, j \in \{0, 1, 2\}$, where $v_f(i)$ denotes the number of edges xy with $f(x)f(y) = i \pmod{3}$. A graph with 3-product cordial labeling is called a 3-product cordial graph. Jeyanthi and Maheswari [4]-[8] proved that the graphs $\langle B_{n,n} : w \rangle, C_n \cup P_n, C_m \circ \overline{K_n}$ if $m \geq 3$ and $n \geq 1$, $P_m \circ \overline{K_n}$ if $m, n \geq 1$, duplicating arbitrary vertex in cycle C_n , duplicating arbitrary edge in cycle C_n , duplicating arbitrary vertex in wheel W_n , middle graph of P_n , the splitting graph of P_n , the total graph of $P_n, P_n[P_2], P_n^2, K_{2,n}$, vertex switching of C_n , ladder L_n , triangular ladder TL_n , the graph $\langle w_n^{(1)} : w_n^{(2)} : \dots : w_n^{(k)} \rangle$, the splitting graphs $S'(K_{1,n}), S'(B_{n,n})$, the shadow graph $D_2(B_{n,n})$, the square graph $B_{n,n}^2$, triangular snake, double alternate triangular snake and alternate triangular snake graphs are 3-product cordial graphs. Also they proved that a complete graph K_n is a 3-product cordial graph if and only if $n \leq 2$.

In addition, they proved that if $G(p, q)$ is a 3-product cordial graph (i)

$p \equiv 1(\text{mod } 3)$ then $q \leq \frac{p^2 - 2p + 7}{3}$. (ii) $p \equiv 2(\text{mod } 3)$ then $q \leq \frac{p^2 - p + 4}{3}$
 (iii) $p \equiv 0(\text{mod } 3)$ then $q \leq \frac{p^2 - 3p + 6}{3}$ and if G_1 is a 3-product cordial graph with $3m$ vertices and $3n$ edges and G_2 is any 3-product cordial graph then $G_1 \cup G_2$ is also 3-product cordial graph. In this paper we establish that switching of an apex vertex in closed helm, double fan, $K_{1,n} \times K_2$ and permutation graph $P(K_2 + mK_1, I)$ are 3-product cordial graphs. We use the following definitions in the subsequent section.

Definition 1.1. The vertex switching G_v of a graph G is the graph obtained by taking a vertex v of G , by removing all the edges incident with v and joining the vertex v to every vertex which is not adjacent to v in G .

Definition 1.2. The helm H_n is the graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex.

Definition 1.3. The closed helm CH_n is the graph obtained from a helm H_n by joining each pendant vertex to form a cycle.

Definition 1.4. The graph $P_n + 2K_1$ is called a double fan DF_n .

Definition 1.5. For any permutation f on $1, 2, 3, \dots, n$, the f -permutation graph on a graph G , $P(G, f)$ consists of two disjoint copies of G , say G_1 and G_2 , each of which has vertices labeled $v_1, v_2, \dots, v_n$ with n edges obtained by joining each v_i in G_1 to $v_{f(i)}$ in G_2 . We denote the identity permutation by I .

For any real number n , $\lceil n \rceil$ denotes the smallest integer $\geq n$ and $\lfloor n \rfloor$ denotes the greatest integer $\leq n$.

2. Main Results

Theorem 2.1. The graph obtained by switching of an apex vertex in closed helm CH_n admits 3-product cordial labeling if and only if $n \equiv 2(\text{mod } 3)$.

Proof. Let v be the apex vertex $v_1, v_2, v_3, \dots, v_n$ be the vertices of inner cycle and $u_1, u_2, u_3, \dots, u_n$ be the vertices of outer cycle CH_n . Let G_v denotes graph obtained by switching of an apex vertex v of $G = CH_n$. Then $|V(G_v)| = 2n + 1$ and $|E(G_v)| = 4n$. We define $f : V(G_v) \rightarrow \{0, 1, 2\}$ as follows:

$$f(v) = 2, \text{ For } 1 \leq i \leq \left\lfloor \frac{2n+1}{3} \right\rfloor, f(v_i) = 0.$$

$$\text{For } n \equiv 0, 2, 3(\text{mod } 4), 1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil,$$

$$f\left(v_{\left\lfloor \frac{2n+1}{3} \right\rfloor + i}\right) = \begin{cases} 1 & \text{for } i \equiv 1, 2(\text{mod } 4) \\ 2 & \text{for } i \equiv 0, 3(\text{mod } 4) \end{cases}$$

$$\text{For } n \equiv 1(\text{mod } 4), n > 5, 1 \leq i \leq \left\lceil \frac{n}{3} \right\rceil - 2,$$

$$f\left(v_{\left\lfloor \frac{2n+1}{3} \right\rfloor + i}\right) = \begin{cases} 1 & \text{for } i \equiv 1, 2(\text{mod } 4) \\ 2 & \text{for } i \equiv 0, 3(\text{mod } 4) \end{cases}$$

$$f(v_{n-1}) = 1 \text{ and } f(v_n) = 2.$$

$$\text{For } n = 5, f(v_4) = 1 \text{ and } f(v_5) = 2.$$

$$\text{For } n \equiv 0, 1, 3(\text{mod } 4), 1 \leq i \leq n, f(u_i) = \begin{cases} 1 & \text{if } i \equiv 1, 2(\text{mod } 4) \\ 2 & \text{if } i \equiv 0, 3(\text{mod } 4) \end{cases}$$

$$\text{For } n \equiv 2(\text{mod } 4), 1 \leq i \leq n-2, f(u_i) = \begin{cases} 1 & \text{if } i \equiv 1, 2(\text{mod } 4) \\ 2 & \text{if } i \equiv 0, 3(\text{mod } 4) \end{cases}$$

$$f(u_{n-1}) = 1 \text{ and } f(u_n) = 2.$$

In view of the above labeling pattern we have $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lceil \frac{2n+1}{3} \right\rceil$, $e_f(0) = e_f(1) = e_f(2) + 1 = \left\lceil \frac{4n}{3} \right\rceil$ if $n \equiv 0, 1(\text{mod } 4)$ and $e_f(0) = e_f(1) + 1 = e_f(2) = \left\lceil \frac{4n}{3} \right\rceil$ if $n \equiv 2, 3(\text{mod } 4)$.

Thus we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence f is a 3-product cordial labeling of G_v if $n \equiv 2(\text{mod } 3)$.

Conversely, we assume that $n \equiv 0(\text{mod } 3)$ and take $n = 3k$. Then $|V(G_v)| = 6k + 1$ and $|E(G_v)| = 12k$.

Let f be a 3-product cordial labeling of G_v . Hence we have $v_f(0) = v_f(1) - 1 = v_f(2) = 2k$ or $v_f(0) = v_f(1) = v_f(2) - 1 = 2k$ and $e_f(0) = e_f(1) = e_f(2) = 4k$. If $f(u_i) = 0$ if $1 \leq i \leq 2k$, then $e_f(0) = 6k + 1$. If $f(v_i) = 0$ if $1 \leq i \leq 2k$, then $e_f(0) = 4k + 1$. If $f(u_i) = 0$ if $1 \leq i \leq 2k - 1$ and $f(v) = 0$, then $e_f(0) = 7k - 1$. If $f(v_i) = 0$ if $1 \leq i \leq 2k - 1$ and $f(v) = 0$, then $e_f(0) = 7k - 1$. Thus, none of $f(u_i), f(v_i)$ and $f(v)$ is 0. From the above argument, we get a contradiction to, f is a 3-product cordial labeling. Hence G_v is not a 3-product cordial graph if $n \equiv 0(\text{mod } 3)$.

We assume that $n \equiv 1(\text{mod } 3)$ and take $n = 3k + 1$. Then $|V(G_v)| = 6k + 3$ and $|E(G_v)| = 12k + 4$.

Let f be a 3-product cordial labeling of G_v . Hence we have $v_f(0) = v_f(1) = v_f(2) = 2k + 1$ and $e_f(0) = 4k + 1$ or $4k + 2$. If $v_f(0) = 2k + 1$, we assign 0 to $2k + 1$ vertices of degree 3. We get $e_f(0) = 4k + 3$. we assign

0 to $2k + 1$ vertices of degree 4. We get $e_f(0) = 6k + 4$. If $f(u_i) = 0$ if $1 \leq i \leq 2k$ and $f(v) = 0$, then $e_f(0) = 7k + 2$. If $f(v_i) = 0$ if $1 \leq i \leq 2k$ and $f(v) = 0$, then $e_f(0) = 7k + 2$. Thus, none of $f(u_i), f(v_i)$ and $f(v)$ is 0. From the above argument, we get a contradiction to, f is a 3-product cordial labeling. Hence G_v is not a 3-product cordial graph if $n \equiv 1(\text{mod } 3)$.

□ An example for the 3-product cordial labeling of a closed helm CH_8 by switching of an apex vertex is shown in Figure 1.

Figure 1

Theorem 2.2. *The double fan graph DF_n is a 3-product cordial graph if and only if $n \equiv 0(\text{mod } 3)$.*

Proof. Let DF_n be the double fan with apex vertices u, v and $v_1, v_2, \dots, v_n$ be the vertices of common path. Then $|V(DF_n)| = n + 2$ and $|E(DF_n)| = 3n - 1$. To define $f : V(DF_n) \rightarrow \{0, 1, 2\}$ as follows:

Let $f(u) = 1, f(v) = 2, f(v_i) = 0$ if $1 \leq i \leq \frac{n}{3}$.

For n is even, $i = \frac{n}{3} + j, 1 \leq j \leq \frac{2n}{3}, f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 2(\text{mod } 4) \\ 2 & \text{if } j \equiv 0, 3(\text{mod } 4) \end{cases}$

For n is odd, $i = \frac{n}{3} + j, 1 \leq j \leq \frac{2n}{3} - 2, f(v_i) = \begin{cases} 1 & \text{if } j \equiv 1, 2(\text{mod } 4) \\ 2 & \text{if } j \equiv 0, 3(\text{mod } 4) \end{cases}$

$f(v_{n-1}) = 1, f(v_n) = 2$.

In view of the above labeling pattern we have $v_f(0) + 1 = v_f(1) = v_f(2) = \frac{n}{3} + 1$ and $e_f(0) = e_f(1) = e_f(2) + 1 = n$ if n is even, $e_f(0) = e_f(1) + 1 = e_f(2) = n$ if n is odd.

Hence, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Thus, f is a 3-product cordial labeling. Therefore, DF_n is a 3-product cordial graph if $n \equiv 0(\text{mod } 3)$.

Conversely, we assume that $n \equiv 2(\text{mod } 3)$ and take $n = 3k + 2$. Then $|V(DF_n)| = 3k + 4$ and $|E(DF_n)| = 9k + 5$.

Let f be a 3-product cordial labeling of DF_n . Hence we have $v_f(0) = v_f(1) - 1 = v_f(2) = k + 1$ or $v_f(0) = v_f(1) = v_f(2) - 1 = k + 1$ and $e_f(0) = 3k + 1$ or $3k + 2$. If $f(u) = 0, f(v) = 0$ and the remaining $k - 1$ vertices are $f(v_i) = 0$ then we get, $e_f(0) > 7k + 3$. If $f(u) = 0$ or $f(v) = 0$ and the remaining k vertices are $f(v_i) = 0$ for $1 \leq i \leq k$ then we get, $e_f(0) = 5k + 2$. If $f(v_i) = 0$ for $1 \leq i \leq k + 1$ then we get $e_f(0) = 3k + 3$. From the above argument, we get a contradiction to, f is a 3-product cordial labeling. Hence DF_n is not a 3-product cordial graph if $n \equiv 2(mod\ 3)$.

We assume that $n \equiv 1(mod\ 3)$ and take $n = 3k + 1$. Then $|V(DF_n)| = 3k + 3$ and $|E(DF_n)| = 9k + 2$.

Let f be a 3-product cordial labeling of DF_n . Hence we have $v_f(0) = v_f(1) = v_f(2) = k + 1$ and $e_f(0) = 3k$ or $3k + 1$. If $f(u) = 0, f(v) = 0$ and the remaining $k - 1$ vertices are $f(v_i) = 0$ then we get, $e_f(0) > 7k + 1$. If $f(u) = 0$ or $f(v) = 0$ and the remaining k vertices are $f(v_i) = 0$ for $1 \leq i \leq k$ then we get, $e_f(0) = 5k + 1$. If $f(v_i) = 0$ for $1 \leq i \leq k + 1$ then we get $e_f(0) = 3k + 3$. From the above argument, we get a contradiction to, f is a 3-product cordial labeling. Hence DF_n is not a 3-product cordial graph if $n \equiv 1(mod\ 3)$. $\square$

An example for the 3-product cordial labeling for the graph DF_6 is shown in Figure 2.

Figure 2.

Theorem 2.3. *The book graph $K_{1,n} \times K_2$ is a 3-product cordial graph.*

Proof. Let the vertices of $K_{1,n} \times K_2$ be $\{u, v, u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n\}$ and the edges are $\{uv\} \cup \{u_i v_i / 1 \leq i \leq n\} \cup \{u u_i / 1 \leq i \leq n\}$. Clearly $K_{1,n} \times K_2$ has $2n + 2$ vertices and $3n + 1$ edges. Define $f : V(K_{1,n} \times K_2) \rightarrow \{0, 1, 2\}$ by the following cases.

$$f(u) = 1, f(v) = 2.$$

Case (i): $n \equiv 0(\text{mod } 3), n = 3k, k > 1$.

$$\text{For } k \text{ is even, } f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 2 & \text{if } k+1 \leq i \leq \frac{3k}{2} \\ 1 & \text{if } \frac{3k}{2} + 1 \leq i \leq 3k \end{cases}$$

$$\text{and } f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k+1 \leq i \leq \frac{3k}{2} \\ 2 & \text{if } \frac{3k}{2} + 1 \leq i \leq 3k \end{cases}$$

$$\text{For } k \text{ is odd, } f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 2 & \text{if } k+1 \leq i < \left\lceil \frac{3k}{2} \right\rceil \\ 1 & \text{if } \left\lceil \frac{3k}{2} \right\rceil \leq i \leq 3k \end{cases}$$

$$\text{and } f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k+1 \leq i < \left\lceil \frac{3k}{2} \right\rceil \\ 2 & \text{if } \left\lceil \frac{3k}{2} \right\rceil \leq i \leq 3k \end{cases}$$

For $k = 1, f(u_1) = f(v_1) = 0, f(u_2) = f(u_3) = 1, f(v_2) = f(v_3) = 2$.

From the above labeling we have, $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \frac{6k+2}{3} \right\rceil$,
 $e_f(0) = e_f(1) = e_f(2) - 1 = 3k$ if k is even and $v_f(0)+1 = v_f(1) = v_f(2) = \left\lceil \frac{6k+2}{3} \right\rceil$,
 $e_f(0) = e_f(1) - 1 = e_f(2) = 3k$ if k is odd.

Case (ii): $n \equiv 1(\text{mod } 3), n = 3k+1, k > 1$.

$$\text{For } k \text{ is even, } f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k+1 \\ 2 & \text{if } k+2 \leq i \leq \left\lceil \frac{3k+1}{2} \right\rceil \\ 1 & \text{if } \left\lceil \frac{3k+1}{2} \right\rceil + 1 \leq i \leq 3k+1 \end{cases}$$

$$\text{and } f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k+2 \leq i \leq \left\lceil \frac{3k+1}{2} \right\rceil \\ 2 & \text{if } i = k+1 \text{ and } \left\lceil \frac{3k+1}{2} \right\rceil + 1 \leq i \leq 3k+1 \end{cases}$$

$$\text{For } k \text{ is odd, } f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k+1 \\ 2 & \text{if } k+2 \leq i \leq \left\lceil \frac{3k+1}{2} \right\rceil \\ 1 & \text{if } \left\lceil \frac{3k+1}{2} \right\rceil + 1 \leq i \leq 3k+1 \end{cases}$$

$$\text{and } f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k \\ 1 & \text{if } k+1 \leq i \leq \left\lceil \frac{3k+1}{2} \right\rceil \\ 2 & \text{if } \left\lceil \frac{3k+1}{2} \right\rceil + 1 \leq i \leq 3k+1 \end{cases}$$

For $k = 1$, $f(u_1) = f(u_2) = f(v_1) = 0$, $f(v_2) = f(u_4) = 1$, $f(u_3) = f(v_3) = f(v_4) = 2$.

From the above labeling we have, $v_f(0) + 1 = v_f(1) = v_f(2) + 1 = \left\lceil \frac{6k+4}{3} \right\rceil$, $e_f(0) - 1 = e_f(1) = e_f(2) = 3k+1$ if k is odd and $v_f(0) + 1 = v_f(1) + 1 = v_f(2) = \left\lceil \frac{6k+4}{3} \right\rceil$, $e_f(0) - 1 = e_f(1) = e_f(2) = 3k+1$ if k is even and $k = 1$.

Case (iii): $n \equiv 2 \pmod{3}$, $n = 3k+2$ and k is even.

$$f(u_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k+1 \\ 2 & \text{if } k+2 \leq i \leq \frac{3k+2}{2} \\ 1 & \text{if } \frac{3k+2}{2} + 1 \leq i \leq 3k+2 \end{cases}$$

$$\text{and } f(v_i) = \begin{cases} 0 & \text{if } 1 \leq i \leq k+1 \\ 1 & \text{if } k+2 \leq i \leq \frac{3k+2}{2} \\ 2 & \text{if } \frac{3k+2}{2} + 1 \leq i \leq 3k+2. \end{cases}$$

From the above labeling we have, $v_f(0) = v_f(1) = v_f(2) = 2k+2$, $e_f(0) - 1 = e_f(1) = e_f(2) = 3k+2$. Hence, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Thus, f is a 3-product cordial labeling. Therefore, $K_{1,n} \times K_2$ is a 3-product cordial graph. $\square$

An example for the 3-product cordial labeling for the book graph $K_{1,7} \times K_2$ is shown in Figure 3.

Figure 3.

Theorem 2.4. *The graph $P(K_2 + mK_1, I)$ is a 3-product cordial graph if and only if $m \equiv 2 \pmod{3}$.*

Proof. Let $V(P(K_2 + mK_1, I)) = \{u, u', v, v', u_i, v_i / 1 \leq i \leq m\}$ and $E(P(K_2 + mK_1, I)) = \{uu_i, u'u_i, vv_i, v'v_i, uu', vv', u'v, uv', u_i v_i / 1 \leq i \leq m\}$. Then $|V(P(K_2 + mK_1, I))| = 2m + 4$ and $|E(P(K_2 + mK_1, I))| = 5m + 4$.

Define a vertex labeling $f : V(P(K_2 + mK_1, I)) \rightarrow \{0, 1, 2\}$ by $f(u) = 1, f(u') = 2, f(v) = 2, f(v') = 1$.

$$f(u_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \\ 1 & \text{for } i = \left\lceil \frac{m}{3} \right\rceil + j \text{ if } j \equiv 1 \pmod{2} \\ 2 & \text{for } i = \left\lceil \frac{m}{3} \right\rceil + j \text{ if } j \equiv 0 \pmod{2} \end{cases}$$

$$\text{and } f(v_i) = \begin{cases} 0 & \text{for } 1 \leq i \leq \left\lceil \frac{m}{3} \right\rceil \\ 1 & \text{for } i = \left\lceil \frac{m}{3} \right\rceil + j \text{ if } j \equiv 0, 3 \pmod{4} \\ 2 & \text{for } i = \left\lceil \frac{m}{3} \right\rceil + j \text{ if } j \equiv 1, 2 \pmod{4}. \end{cases}$$

In view of the above labeling pattern, we have $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lceil \frac{2m+4}{3} \right\rceil, e_f(0) = e_f(1) + 1 = e_f(2) = \left\lceil \frac{5m+4}{3} \right\rceil$ if m is even and $v_f(0) + 1 = v_f(1) = v_f(2) = \left\lceil \frac{2m+4}{3} \right\rceil, e_f(0) = e_f(1) = e_f(2) + 1 = \left\lceil \frac{5m+4}{3} \right\rceil$ if m is odd. Thus, we have $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_f(j)| \leq 1$ for all $i, j = 0, 1, 2$. Hence, f is a 3-product cordial labeling of $P(K_2 + mK_1, I)$ if $m \equiv 2 \pmod{3}$.

Conversely, we assume that $m \equiv 0 \pmod{3}$ and take $m = 3k$. Then $|V(P(K_2 + mK_1, I))| = 6k + 4$ and $|E(P(K_2 + mK_1, I))| = 15k + 4$. Let f be a vertex 3-product cordial labeling of $P(K_2 + mK_1, I)$. Then $v_f(0) - v_f(1) - 1 = v_f(2) = 2k + 1$ or $v_f(0) = v_f(1) = v_f(2) - 1 = 2k + 1$ and $e_f(0) = 5k + 1$ or $5k + 2$. If $f(u), f(u'), f(v), f(v')$ are zero and the remaining $2k - 3$ vertices are either $f(u_i) = 0$ for $1 \leq i \leq k - 2$, $k > 2$ and $f(v_i) = 0$ for $1 \leq i \leq k - 1$, $k > 2$ or $f(u_i) = 0$ for $1 \leq i \leq k - 1$, $k > 2$ and $f(v_i) = 0$ for $1 \leq i \leq k - 2$, $k > 2$ then $e_f(0) = 13k + 3$. If one of $f(u)$ or $f(u')$ and $f(v)$ or $f(v')$ is zero and the remaining $2k - 1$ vertices are either $f(u_i) = 0$ for $1 \leq i \leq k - 1$, $k > 1$ and $f(v_i) = 0$ for $1 \leq i \leq k$, $k > 1$ or $f(u_i) = 0$ for $1 \leq i \leq k$, $k > 1$ and $f(v_i) = 0$ for $1 \leq i \leq k - 1$, $k > 1$ then $e_f(0) = 7k + 3$ or $7k + 4$. If all the $2k + 1$ vertices are either $f(u_i) = 0$ for $1 \leq i \leq k + 1$ and $f(v_i) = 0$ for $1 \leq i \leq k$ or $f(u_i) = 0$ for $1 \leq i \leq k$ and $f(v_i) = 0$ for $1 \leq i \leq k + 1$ then $e_f(0) = 5k + 3$. Hence none of $f(u), f(u'), f(v)$ and $f(v')$ is zero. From the above arguments, we get a contradiction to, f is a 3-product cordial labeling. Hence, $P(K_2 + mK_1, I)$ is not a 3-product cordial graph if $m \equiv 0 \pmod{3}$.

We assume that $m \equiv 1 \pmod{3}$ and take $m = 3k + 1$. Then $|V(P(K_2 + mK_1, I))| = 6k + 6$ and $|E(P(K_2 + mK_1, I))| = 15k + 9$. Let f be a vertex 3-product cordial labeling of $P(K_2 + mK_1, I)$. Hence we have $v_f(0) = v_f(1) = v_f(2) = 2k + 2$ and $e_f(0) = e_f(1) = e_f(2) = 5k + 3$. If $f(u), f(u'), f(v), f(v')$ are zero and the remaining $2k - 2$ vertices are $f(u_i) = f(v_i) = 0$ for $1 \leq i \leq k - 1$, $k > 1$ then $e_f(0) = 13k + 7$. If one of $f(u)$ or $f(u')$ and $f(v)$ or $f(v')$ is zero and the remaining $2k$ vertices are $f(u_i) = f(v_i) = 0$ for $1 \leq i \leq k$ then $e_f(0) = 9k + 6$ or $9k + 5$. If all the $2k + 2$ vertices are $f(u_i) = f(v_i) = 0$ for $1 \leq i \leq k + 1$ then $e_f(0) = 5k + 5$. Hence none of $f(u), f(u'), f(v)$ and $f(v')$ is zero. From the above arguments, we get a contradiction to, f is a 3-product cordial labeling. Hence, $P(K_2 + mK_1, I)$ is not a 3-product cordial graph if $m \equiv 1 \pmod{3}$. $\square$

An example for the 3-product cordial labeling for the graph $P(K_2 + 5K_1, I)$ is shown in Figure 4.

Figure 4.

References

- [1] I. Cahit, Cordial Graphs: A weaker version of graceful and harmonious graphs, *Ars Combinatoria*, **23**, pp. 201-207, (1987).
- [2] Joseph A. Gallian, A dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, (2018) # DS6.
- [3] F. Harary, *Graph Theory*, Addison Wesley, Massachusetts, (1972).
- [4] P. Jeyanthi and A. Maheswari, 3-product cordial labeling of some graphs, *International Journal on Mathematical Combinatorics*, **1**, pp. 96-105, (2012).
- [5] P. Jeyanthi and A. Maheswari, 3-product cordial labeling, *SUT Journal of Mathematics*, **48**, pp. 231-240, (2012).
- [6] P. Jeyanthi and A. Maheswari, 3-product cordial labeling of star graphs, *Southeast Asian Bulletin of Mathematics*, **39**, pp. 429-437, (2015).
- [7] P. Jeyanthi and A. Maheswari, Some results on 3-product cordial labeling, *Utilitas Mathematica*, **99**, pp. 215-229, March, (2016).

- [8] P. Jeyanthi, A. Maheswari and M. Vijayalakshmi, 3-Product cordial labeling of some snake graphs, *Proyecciones Journal of Mathematics*, **38**(1), Vol. 38, No. 1, pp. 13-30, March, (2019).
- [9] R. Ponraj, M. Sivakumar and M. Sundaram, k -product cordial labeling of graphs, *Int.J. Contemp. Math. Sciences*, **7**, (15), pp. 733-742, (2012).
- [10] M. Sundaram, R. Ponraj and S. Somasundaram, Product Cordial labeling of graphs, *Bulletin of Pure and Applied Sciences*, **23E**(1), pp. 155-163, (2004).
- [11] M. Sundaram, R. Ponraj and S. Somasundaram, EP -cordial labeling of graphs, *Varahmihir Journal of Mathematical Sciences*, **7**(1), pp. 183-194, (2007).

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