On the (M,D) number of a graph

J. John
Government College of Engineering, India

P. Arul Paul Sudhahar
Rani Anna Gobernment Constituent College for Women, India

and

D. Stalin
Bharathiyar University, India

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Abstract

For a connected graph $G = (V,E)$, a monophonic set of $G$ is a set $M \subseteq V(G)$ such that every vertex of $G$ is contained in a monophonic path joining some pair of vertices in $M$. A subset $D$ of vertices in $G$ is called dominating set if every vertex not in $D$ has at least one neighbour in $D$. A monophonic dominating set $M$ is both a monophonic and a dominating set. The monophonic,dominating,monophonic domination number $m(G), \gamma(G), \gamma_m(G)$ respectively are the minimum cardinality of the respective sets in $G$. Monophonic domination number of certain classes of graphs are determined. Connected graph of order $p$ with monophonic domination number $p - 1$ or $p$ is characterised. It is shown that for every two integers $a, b \geq 2$ with $2 \leq a \leq b$, there is a connected graph $G$ such that $\gamma_m(G) = a$ and $\gamma_g(G) = b$, where $\gamma_g(G)$ is the geodetic domination number of a graph.

Keywords: monophonic number, domination number, monophonic domination number, geodetic domination number.

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1. Introduction

By a graph $G=(V,E)$, we mean a finite undirected graph without loops or multiple edges. The order and size of $G$ are denoted by $p$ and $q$ respectively. For basic graph theoretic terminology we refer to Harary [1]. The vertices $u$ and $v$ in a connected graph $G$, the distance $d(u,v)$ is the length of a shortest $u-v$ path in $G$. The eccentricity $e(v)$ of a vertex $v$ in $G$ is the maximum distance from $v$ and a vertex of $G$. The minimum eccentricity among the vertices of $G$ is the radius, $\text{rad}(G)$ or $r(G)$ and the maximum eccentricity is its diameter, $\text{diam}(G)$ of $G$. An $u-v$ path of length $d(u,v)$ is called an $u-v$ geodesic. A vertex $x$ is said to lie on a $u-v$ geodesic $P$ if $x$ is a vertex of $P$ including the vertices $u$ and $v$. A geodesic set of $G$ is a set $S \subseteq V(G)$ such that every vertex of $G$ contained in a geodesic joining some pair of vertices in $S$. The geodetic number $g(G)$ of $G$ is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a geodetic basis of $G$. The geodetic number was introduced in [7] and further studied in [4,8].

A chord of a path $P$ is an edge joining two non-adjacent vertices of $P$. A path $P$ is called monophonic if it is a chordless path. A monophonic set of $G$ is set $M \subseteq V$ such that every vertex of $G$ is contained in a monophonic path joining some pair of vertices in $M$. The monophonic number $m(G)$ of $G$ is the minimum order of its monophonic sets and any monophonic set of order $m(G)$ is a minimum monophonic set or simply a $m-$set of $G$. The monophonic number of a graph $G$ is studied in [5,6,9]. If $e=uv$ is an edge of a graph $G$ with $d(u)=1$ and $d(v)>1$, then we call $e$ a pendent edge, $u$ a leaf and $v$ a support vertex. Let $L(G)$ be the set of all leaves of a graph $G$. We denote by $P_p, C_p$ and $K_{r,s}$, the path on $p$ vertices, the cycle on $p$ vertices and complete bipartite graph in which one partite set has $r$ vertices and the other partite set has $s$ vertices respectively. For any set $M$ of vertices of $G$, induced subgraph $<M>$ is the maximal subgraph of $G$ with vertex set $M$. For any connected graph $G$, a vertex $v \in V(G)$ is called a cut vertex of $G$ if $<V-\{v\}>$ is no longer connected. A maximum connected induced subgraph without a cut vertex is called a block of $G$. A graph $G$ is a block graph if every block in $G$ is complete. Sum of two graphs $G_1$ and $G_2$ is the union of $G_1$ and $G_2$ together with all the lines joining vertices of $G_1$ to vertices of $G_2$. Let $N(v) = \{u \in V(G) : uw \in E(G)\}$ is called the neighborhood of the vertex $v$ in $G$. A vertex $v$ is a simplicial vertex of a graph $G$ if $N(v)$ is complete. A simplex of a graph $G$ is a subgraph of $G$ which is a complete graph. A vertex $v$ in a graph $G$ dominates itself and its neighbors. A set of vertices $D$ in a graph $G$ is a
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A dominating set if each vertex of $G$ is dominated by some vertices of $D$. The dominating number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. For references on domination parameters in graphs see [2,3]. A set of vertices $M$ in $G$ is called a geodetic dominating set if $M$ is both a geodetic set and a dominating set. The minimum cardinality of a geodetic dominating set of $G$ is its geodetic domination number and is denoted by $\gamma_g(G)$. A geodetic dominating set of size $\gamma_g(G)$ is said to be a $\gamma_g$-set. The geodetic domination number of a graph was introduced and studied in [8]. It is easily seen that a dominating set is not in general a monophonic set in a graph $G$. Also the converse is not valid in general. This has motivated us to study the new domination conception of monophonic domination. We investigate subsets of vertices of a graph that are both a monophonic set and a dominating set. We call these sets as a monophonic dominating sets. We call the minimum cardinality of the monophonic dominating set of $G$, the monophonic domination number of $G$. Throughout this paper $G$ denotes simple connected graph with at least two vertices.

The following theorems are used in sequel.

**Theorem 1.1.** [9] Each simplicial vertex of a connected graph $G$ belongs to every monophonic set of $G$. In particular every end vertex of a connected graph $G$ belongs to every monophonic set of $G$.

**Theorem 1.2.** [8] Each simplicial vertex of a connected graph $G$ belongs to every geodetic dominating set of $G$. In particular every end vertex of a connected graph $G$ belongs to every geodetic dominating set of $G$.

2. The Monophonic Domination Number Of a Graph

**Definition 2.1.** Let $G$ be a connected graph. A set of vertices $M$ in $G$ is called a monophonic dominating set or simply $(M,D)$-set if $M$ is both a monophonic set and a dominating set. The minimum cardinality of a $(M,D)$-set of $G$ is its monophonic domination number or simply $(M,D)$-number and is denoted by $\gamma_m(G)$. A $(M,D)$-set of size $\gamma_m(G)$ is said to be a $\gamma_m$-set.

**Example 2.2.** For the graph $G$ is given in Figure 2.1, $M = \{v_1, v_4\}$ is a $(M,D)$-set of $G$ so that $\gamma_m(G) = 2$. 
Remark 2.3. Each simplicial vertex of a connected graph $G$ belongs to every $(M, D)$-set of $G$.

Remark 2.4. Let $G$ be a connected graph and $v$ be a cut-vertex of $G$. Then every $(M, D)$-set contains at least one element from each component of $G - v$.

Remark 2.5. If $G$ is a connected graph of order $p$, then
\[ 2 \leq \max\{m(G), \gamma(G)\} \leq \gamma_m(G) \leq p. \]

Remark 2.6. For any cycle $C_p$, $(p \geq 4)$, \( \gamma_m(C_p) = \gamma(C_p) = \lceil p/3 \rceil \).

In the following, we determine the $(M,D)$-number of some standard graphs.

Theorem 2.7. For the complete graph $K_p$ ($p \geq 2$), \( \gamma_m(K_p) = p \).

Proof. Since every vertex of the complete graph $K_p$ ($p \geq 2$) is a simplical vertex, the vertex set of $K_p$ is the unique $(M, D)$-set of $K_p$. Thus \( \gamma_m(K_p) = p \).

Theorem 2.8. For the wheel $G = W_p$ ($p \geq 4$),
\[ \gamma_m(w_p) = \begin{cases} 4, & \text{if } p = 4; \\ 2, & \text{if } p = 5, 6; \\ 3, & \text{if } p \geq 7. \end{cases} \]
Proof. Let \( \{x,v_1,v_2...v_{p-1}\} \) be the vertices of \( G=W_p(p \geq 4) \), with \( \text{deg}(x) = p-1 \).

Case(i) Let \( p = 4 \). Then \( G = K_4 \) and by Theorem 2.7, \( \gamma_m(W_p) = 4 \).

Case(ii) Let \( p = 5 \) or \( 6 \). Then \( M = \{v_1,v_3\} \) is a \((M,D)\)-set of \( G \) so that \( \gamma_m(W_p) = 2 \).

Case(iii) Let \( p \geq 7 \). Let \( M = \{x,v_i,v_j\} \) \((1 \leq i \neq j \leq p-1)\), where \( v_i \) and \( v_j \) are any two non adjacent vertices of \( G \). Then \( M \) is a \((M,D)\)-set of \( G \) so that \( \gamma_m(G) \leq 3 \). Suppose that \( \gamma_m(G) = 2 \). Then there exists a \((M,D)\)-set \( M' \) such that \(|M'| = 2 \). If \( M' = \{x,v_i\}, \{1 \leq i \leq p-1\} \), then \( xv_i \), \((1 \leq i \leq p-1)\) is a chord of path \( x-v_i \) and so \( M' \) is not a \((M,D)\)-set of \( G \), which is a contradiction. If \( M' = \{v_i,v_j\} \) \((1 \leq i \neq j \leq p-1)\) then \( M' \) is a monophonic set of \( G \) which is not a dominating set of \( G \), which is a contradiction. Therefore \( \gamma_m(W_p) = 3 \). \( \square \)

Theorem 2.9. For the complete bipartite graph \( G = K_{m,n}, \gamma_m(K_{m,n}) = 2 \), if \( m = n = 1 \)
\( n \) if \( n \geq 2, m = 1 \)
\( \min\{m,n,4\} \) if \( m,n \geq 2 \).

Proof. Case(i). Let \( m=n=1 \). Then \( G = K_2 \). By Theorem 2.7, \( \gamma_m(G) = 2 \).

Case(ii). Let \( m = 1, n \geq 2 \). Then \( G = K_{1,n} \). Let \( M \) be the set of \( n \) end vertices of \( G \). Then by Remark 2.3, \( \gamma_m(G) \geq n \). It is clear that \( M \) is a \((M,D)\)-set of \( G \) so that \( \gamma_m(G) = n \).

Case(iii) Let \( 2 \leq m \leq n \). Let \( U = \{u_1, u_2...u_m\} \) and \( V = \{v_1, v_2...v_n\} \) be the bipartite sets of \( G \).

Subcase iii(a). Let \( m = 2, n \geq 2 \). Then \( U = \{u_1, u_2\} \) is a \((M,D)\)-set of \( G \) so that \( \gamma_m(G) = 2 \).

Subcase iii(b). Let \( m = 3 \) and \( n \geq 3 \). Then \( M = \{u_1, u_2, u_3\} \) is a \((M,D)\)-set of \( G \) and so \( \gamma_m(G) \leq 3 \). Let \( M' \) be a \((M,D)\)-set of \( G \) with \(|M'| = 2 \). If \( M' \subset U \), then there exists \( x \in U \) such that \( x \notin M' \). Then the vertex \( x \) does not lie on a monophonic path joining a pair of vertices of \( M' \), which is a contradiction. If \( M' \subset W \), then there exists at least one \( y \in W \) such that \( y \notin M' \). Then the at least one \( y \notin M' \). Then the the vertex \( y \) does not lie on a monophonic path joining a pair of vertices of \( M' \), which is a contradiction. If \( M' \subset U \cup W \), then \( M' = \{u_i, w_j\} \) \((1 \leq i \leq 3), (1 \leq j \leq n)\). Since \( u_i, w_j \) is a chord of the path \( u_i - w_j \), \( M' \) is not a \((M,D)\)-set of \( G \), which is a contradiction. Therefore \( \gamma_m(G) = 3 \).
Subcase iiic. Let $m \geq 4$ and $n \geq 4$. Then $M = \{u_1, u_2, v_1, v_2\}$ is a $(M, D)$ set of $G$ and so that $\gamma_m(G) \leq 4$. By the similar argument given in Subcase iiib, there is no $(M, D)$-set $M'$ such that $|M'| = 2$ or $|M'| = 3$. Hence $\gamma_m(G) = 4$. 

**Theorem 2.10.** If $G$ is a non complete connected graph such that it has a minimum cut set, then $\gamma_m(G) \leq p - k(G)$.

**Proof.** Since $G$ is non complete, it is clear that $1 \leq k(G) \leq p - 2$. Let $U = \{u_1, u_2, ..., u_k\}$ be a minimum cut set of $G$. Let $G_1, G_2, ..., G_r (r \geq 2)$ be the components of $G - U$ and let $M = V(G) - U$. Then every vertex $u_i (1 \leq i \leq k)$ is adjacent to at least one vertex of $G_j$ for every $j (1 \leq j \leq r)$. It is clear that $M$ is a $(M, D)$-set of $G$ so that $\gamma_m(G) \leq p - k(G)$. 

**Theorem 2.11.** Let $G$ be a connected graph of order $p \geq 2$. Then $\gamma_m(G) = 2$ if and only if there exist a $(M, D)$-set $M = \{u, v\}$ of $G$ such that $d(u, v) \leq 3$.

**Proof.** Suppose $\gamma_m(G) = 2$. Let $M = \{u, v\}$ be a $(M, D)$-set of $G$. Suppose that $d(u, v) \geq 4$. Then the diametrical path contains at least three internal vertices. Therefore $\gamma_m(G) \geq 3$, which is a contradiction. Therefore $d(u, v) \geq 3$. The converse is clear. 

**Theorem 2.12.** Let $G$ be a connected graph of order $p \geq 2$. Then $\gamma_m(G) = p$ if and only if $G$ is the complete graph on $p$ vertices.

**Proof.** Suppose $G = K_p$. Then by Theorem 2.7, $\gamma_m(G) = p$. Conversely, let $\gamma_m(G) = p$. Suppose that $G$ is non complete. Then by Theorem 2.10, $\gamma_m(G) \leq p - 1$, which is a contradiction. It follows that $G$ is complete. 

**Theorem 2.13.** Let $G$ be a connected graph of order $p \geq 2$. Then $\gamma_m(G) = p - 1$ if and only if $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geq 2, j \geq 1$.

**Proof.** Suppose $\gamma_m(G) = p - 1$. Then by Theorem 2.10, $k(G) = 1$. Therefore $G$ contains only one cut vertex, say $v$. We show that each component of $G - \{v\}$ is complete. Suppose that there exist a component $G_1$ of $G - \{v\}$ such that $G_1$ is non complete. Then $|G_1| \geq 2$. Let $u$ be the non simplicial vertex of $G_1$. Then $M = V(G) - \{u, v\}$ is a $(M, D)$-set of $G$ so that $\gamma_m(G) \leq p - 2$, which is a contradiction. Hence each component of $G - \{v\}$ is complete. Therefore $G = K_1 + \bigcup m_j K_j$, where $\sum m_j \geq 2$. Conversely suppose $G = K_1 + \bigcup m_j K_j$ where $\sum m_j \geq 2$. Then it is clear that $\gamma_m(G) = p - 1$. 

**Remark 2.14.** If $G$ is a graph of order $p$, then $\gamma_m(G) + \gamma_m(\bar{G}) \leq 2p$ and $\gamma_m(G) + \gamma_m(\bar{G}) = 2p$ if and only if $G = K_p$ or $G = \bar{G} = K_p$.

**Theorem 2.15.** If $G$ is a connected graph of order $p$, then $\gamma_m(G) + \gamma_m(\bar{G}) = 2p - 1$ if and only if $p \geq 3$ and $G = K_{1, p-1}$ or $G = K_{1, p-1}$. 

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Proof. Suppose \( p \geq 3 \) and \( G = K_{1,p-1} \) or \( \overline{G} = K_{1,p-1} \). Then by Theorem 2.9(ii), \( \gamma_m(G) + \gamma_m(\overline{G}) = 2p - 1 \). Conversely suppose \( \gamma_m(G) + \gamma_m(\overline{G}) = 2p - 1 \). Then \( \gamma_m(G) = p \) or \( \gamma_m(\overline{G}) = p \). Without loss of generality, we assume that \( \gamma_m(G) = p \). Then \( \gamma_m(G) = p - 1 \). We prove that the components of \( G \) are complete graphs. If not, then \( G \) contains a component \( H \) with two non adjacent vertices \( u \) and \( v \). Let \( P \) be a path in \( u - v \) geodesic in \( H \) and \( x \) be a vertex of \( P \) adjacent to \( u \). Let \( S = V(G) - \{x\} \). Then \( S \) is a monophonic dominating set of \( G \) so that \( \gamma_m(G) \leq p - 1 \), which is a contradiction to \( \gamma_m(\overline{G}) = p \). If \( G \) is not connected, then \( p \geq 2 \) and \( G \) is connected. By Theorem 2.13, we find that there exists a vertex \( v \) in \( G \) such that \( v \) is adjacent to every other vertex of \( G \) and \( G - v \) is the union of at least two complete graphs. Therefore \( p \geq 3 \). Since \( \gamma_m(\overline{G}) = p \), the components of \( G - \{v\} \) are isolated vertices. This shows that \( G = K_{1,p-1} \). \( \square \)

3. Realization results

**Theorem 3.1.** For any two integers \( a,b \geq 2 \), there is a connected graph \( G \) such that \( \gamma(G) = a, m(G) = b \) and \( \gamma_m(G) = a + b \).

Proof. Let \( F : r,s,u,v,t,r \) be a copy of \( C_5 \). Let \( H \) be a graph obtained from \( F \) by adding the new vertices \( z_1, z_2, ..., z_{b-1} \) and join each to the vertex \( r \). Let \( G \) be the graph obtained from \( H \) by taking a copy of the path on \( 3(a-2)+1 \) vertices \( y_0, y_1, ..., y_{3(a-2)} \) and joining \( y_0 \) to the vertex \( u \) as shown in the Figure 3.1. Let \( Z = \{r, u, y_2, y_3, ..., y_{3(a-2)}\} \). Then it is clear that \( Z \) is a minimum dominating set of \( G \) so that \( \gamma(G) = a \). Let \( Z' = \{z_1, z_2, ..., z_{b-1}, y_3(a-2)\} \). Then by Theorem 1.1, \( Z' \) is a subset of every monophonic set of \( G \) and so \( m(G) \geq b \). Now \( Z' \) is a monophonic set of \( G \) so that \( m(G) = b \). By Remark 2.3, \( Z' \) is subset of every \((M,D)\)-set of \( G \). Now, let \( M = Z \cup Z' \). It is clear that \( M \) is a minimum \((M,D)\)-set of \( G \) so that \( \gamma_m(G) = a + b \). \( \square \)
Theorem 3.2. For any two integers \( a, b \geq 2 \) with \( 2 \leq a \leq b \), there is a connected graph \( G \) such that \( \gamma_m(G) = a \) and \( \gamma_g(G) = b \).

**Proof.** Let \( P : x, y, z \) be a path on three vertices. Let \( P_i : u_i, v_i(1 \leq i \leq b - a + 2) \) be a path on two vertices. Let \( H \) be a graph obtained from \( P \) and \( P_i \) by joining each \( u_i(1 \leq i \leq b - a + 2) \) with \( x \) and each \( v_i(1 \leq i \leq b - a + 2) \) with \( z \). Let \( G \) be a graph obtained from \( H \) by adding the new vertices \( z_1, z_2, \ldots, z_{a-2} \) and joining each \( z_i(1 \leq i \leq a - 2) \) with \( x \) and \( y \) as shown in Figure 3.2. First we show that \( \gamma_m(G) = a \). Let \( Z' = \{z_1, z_2, \ldots, z_{a-2}\} \) be the set of all of simplicial vertices of \( G \). By Remark 2.3, \( Z \) is subset of every \((M,D)\)-set of \( G \). It is clear that \( Z \) is not a \((M,D)\)-set of \( G \) and so \( \gamma_m(G) \geq a \). However \( M = Z \cup \{x, z\} \) is a \((M,D)\)-set of \( G \) so that \( \gamma_m(G) = a \). Next, we show that \( \gamma_g(G) = b \). By Theorem 1.2 \( Z \) is subset of every geodetic dominating.
set of $G$. It can be easily verified that $Z$ is not a geodetic dominating set of $G$. Now $M = Z \cup \{v_1, v_2, \ldots, v_{b-a+2}\}$ is a geodetic dominating set of $G$ so that $\gamma_g(G) \leq b$. Let $H_i = \{u_i, v_i\} (1 \leq i \leq b - a + 2)$. Let $S$ be a geodetic dominating set of $G$. Suppose that $z \in S$. Then $S$ contains at least one element of each $H_i (1 \leq i \leq b - a + 2)$. If not suppose that $u_1, v_1 \notin S$. Then $u_1, v_1$ do not lie on a geodesic joining a pair of vertices of $S$, which is a contradiction. Therefore $S$ contains at least one element of each $H_i (1 \leq i \leq b - a + 2)$. Hence it follows that $\gamma_g(G) \geq a - 2 + 1 + b - a + 2 = b + 1$, which is a contradiction. Therefore $z \notin S$. Let $G_i = \{u_i, v_{i+1}\} (1 \leq i \leq b - a + 1)$, $Q_i = \{v_i, u_{i+1}\} (1 \leq i \leq b - a + 1)$ and $S' = \{v_1, v_2, \ldots, v_{b-a+2}\}$. It is easily observed that $S$ contain at least one element from each $G_i (1 \leq i \leq b - a + 2)$ or least one element from each $Q_i (1 \leq i \leq b - a + 2)$ or $S' \subseteq S$. Hence it follows that $\gamma_g(G) = a - 2 + b - a + 2 = b$. □

4. Block Graphs

**Theorem 4.1.** Let $G$ be a connected block graph of order $p \geq 2$, and let $M$ be the set of simplicial vertices of $G$. Then $M$ is the unique minimum monophonic set of $G$.

**Proof.** The theorem is obvious when $M = V(G)$. Hence assume that $M \subset V(G)$. Let $v \in V(G) - M$ be an arbitrary vertex. It follows that $v$ is a cut-vertex of $G$. Let $H$ and $H'$ be two components of $G - \{v\}$ and $H$ and $H'$ are also block
graphs. Let \( x \in V(H) \) and \( x' \in V(H') \) be two simplicial vertices of \( G \). Let \( P \) be a monophonic path from \( x \) to \( x' \) in \( G \). Since \( v \) is a cut-vertex of \( G \) containing \( v \), the monophonic path contains \( v \). Hence \( P \) is a \( x - x' \) monophonic path of \( G \) containing \( v \). Then \( J[M] = V(G) \). Thus \( M \) is a monophonic set of \( G \). As every monophonic set \( M' \) of \( G \) must contain \( M \), the set \( M \) is the unique monophonic set of \( G \). \end{proof}

**Theorem 4.2.** If \( G \) is a connected block graph of order \( p \geq 2 \), then the following conditions are equivalent.

\[(a) \; \gamma_m(G) = m(G) = \gamma(G).
\]

\[(b) \; \text{The set } M \text{ of simplicial vertices of } G \text{ is a minimum dominating set of } G.
\]

\[(c) \; \text{Every block of } G \text{ contains at most one simplicial vertex, and every vertex of } G \text{ belongs to exactly one simplex of } G.
\]

**Proof.** \((a) \Rightarrow (b)\). Suppose \( \gamma_m(G) = m(G) = \gamma(G) \). Then by Theorem 4.1, the set \( M \) of simplicial vertices of \( G \) is a minimum dominating set of \( G \).

\((b) \Rightarrow (a)\). Suppose the set \( M \) of simplicial vertices of \( G \) is a minimum dominating set of \( G \). It follows that \( \gamma_m(G) = m(G) = \gamma(G) \).

\((c) \Rightarrow (b)\). Let \( G_1, G_2, \ldots, G_k \) be the simplexes of \( G \) with simplicial vertices \( v_i \in V(G_i) \) for \( i = \{1, 2, \ldots, k\} \). Clearly each simplex \( G_i \) is also a block of \( G \). Since every block of \( G \) contains at most one simplicial vertex, \( v_i \) is the only simplicial vertex of \( G_i \). The hypothesis that every vertex of \( G \) belongs to exactly one simplex of \( G \) shows that \( V(G) = V(G_1) \cup V(G_2) \cup \ldots \cup V(G_k) \). Therefore \( M = \{v_1, v_2, \ldots, v_k\} \) is a dominating set of \( G \). On the contrary, suppose that \( G \) contains a dominating set \( M' \) with \( |M'| < |M| \). This implies that there exists a vertex \( y \in M' \) such that \( y \) dominates at least two simplicial vertices, say \( v_1 \) and \( v_2 \) which is a contradiction. This contradiction shows that \( y \) belongs to the simplexes \( G_1 \) and \( G_2 \). Hence \( M \) is a minimum dominating set of \( G \).

\((b) \Rightarrow (c)\). Suppose that the set \( M \) of simplicial vertices of \( G \) is a minimum dominating set of \( G \). If there is a block containing two simplicial vertices \( u \) and \( v \), then \( M - \{u\} \) is also a dominating set of \( G \), which is a contradiction. This shows that every block of \( G \) contains at most one simplicial vertex. If there exists a vertex which does not belong to any simplex of \( G \), then \( M \) is not a dominating set of \( G \), which is a contradiction. Finally, on the contrary, suppose that there is a vertex \( u \) belonging to at least two simplexes of \( G_1 \) and \( G_2 \). If \( v_1 \) and \( v_2 \) are simplicial vertices of \( G_1 \) and \( G_2 \), then \( (M - \{v_1, v_2\}) \cup \{u\} \) is a dominating set of \( G \), which is a contradiction. Hence, every vertex of \( G \) belong to exactly one simplex of \( G \). \end{proof}
References


J. John
Department of Mathematics, 
Government College of Engineering, 
Tirunelveli-627 007, 
India 
e-mail: john@gcetly.ac.in

P. Arul Paul Sudhahar
Department of Mathematics
Rani Anna Government constituent college for women
Tirunelveli-627 008
e-mail:arulpaulsudhahar@gmail.com

and
D.S. Stalin  
Research and Development Center  
Bharathiyar University  
Coimbatore-641 046  
e-mail: stalinndd@gmail.com