Total domination and vertex-edge domination in trees

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Abstract

A vertex v of a graph $G = (V, E)$ is said to *ve-dominate* every edge incident to v , as well as every edge adjacent to these incident edges. A set $S \subseteq V$ is a *vertex-edge dominating set* if every edge of E is *ve-dominated* by at least one vertex of S . The minimum cardinality of a vertex-edge dominating set of G is the *vertex-edge domination number* $\gamma_{ve}(G)$. In this paper we prove $(\gamma_t(T) - \ell + 1)/2 \leq \gamma_{ve}(T) \leq (\gamma_t(T) + \ell - 1)/2$ and characterize trees attaining each of these bounds.

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1. Introduction

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex v of G we mean the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The degree of a vertex v , denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. The path on n vertices we denote by P_n . Let T be a tree, and let v be a vertex of T . We say that v is adjacent to a path P_n if there is a neighbor of v , say x , such that the subtree resulting from T by removing the edge vx and which contains the vertex x as a leaf, is a path P_n .

A subset $D \subseteq V(G)$ is a dominating set, abbreviated DS, of G if every vertex of $V(G) \setminus D$ has a neighbor in D . The domination number of a graph G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G . A subset $D \subseteq V(G)$ is a total dominating set, abbreviated TDS, of G if every vertex of $V(G)$ has a neighbor in D . The total domination number of a graph G , denoted by $\gamma_t(G)$, is the minimum cardinality of a total dominating set of G . A total dominating set of G of minimum cardinality is called a $\gamma_t(G)$ -set. For more details on total domination, see [2].

An edge $e \in E(G)$ is vertex-edge dominated (ve -dominated) by a vertex $v \in V(G)$ if e is incident to v , or e is adjacent to an edge incident to v . A subset $D \subseteq V(G)$ is a vertex-edge dominating set, abbreviated VEDS, of G if every edge of G is vertex-edge dominated by a vertex of D . The vertex-edge domination number of G , denoted by $\gamma_{ve}(G)$, is the minimum cardinality of a vertex-edge dominating set of G . A vertex-edge dominating set of G of minimum cardinality is called a $\gamma_{ve}(G)$ -set. Vertex-edge domination in graphs was introduced in [5], and further studied in [1, 3, 4].

In [4], trees with equal domination number and vertex-edge domination number are characterized. Here, we prove $(\gamma_t(T) - \ell + 1)/2 \leq \gamma_{ve}(T) \leq (\gamma_t(T) + \ell - 1)/2$ and characterize trees attaining each of these bounds.

2. Main Results

The one vertex graph does not have total dominating set and vertex-edge dominating set, in this paper, by a tree we mean only a connected graph with no cycle, and which has at least two vertices.

We begin with the following observations:

Observation 1. *Every support vertex of a graph G is in every TDS of graph G .*

Observation 2. Let G be a graph and $u \in V(G)$. Let the vertex u be adjacent to two paths vw and xy . Let v and x be adjacent to u . Let $H = G - \{x, y\}$. Then $\gamma_t(G) = \gamma_t(H) + 1$ and $\gamma_{ve}(G) = \gamma_{ve}(H)$.

Proof. Let D be a $\gamma_t(G)$ -set. By observation 1, the vertices $v, x \in D$. To dominate the vertices v and x , the vertex $u \in D$. It is easy to see that $D \setminus \{x\}$ is a TDS of the graph H . Thus $\gamma_t(H) \leq \gamma_t(G) - 1$. Let D' be a $\gamma_t(H)$ -set. By observation 1, the vertex $v \in D'$. To dominate v , the vertex $u \in D'$. It is clear that $D' \cup \{x\}$ is a TDS of the graph G . Thus $\gamma_t(G) \leq \gamma_t(H) + 1$. We get $\gamma_t(G) = \gamma_t(H) + 1$.

Let D be a $\gamma_{ve}(G)$ -set. To dominate the edges xy and vw , the vertex $u \in D$. Obviously D is a VEDS of the graph H . Thus $\gamma_{ve}(H) \leq \gamma_{ve}(G)$. Let D' be a $\gamma_{ve}(H)$ -set. To dominate the edge vw , the vertex $u \in D'$. Clearly the vertex u dominates the edges ux and xy in the graph G . The set D' is a VEDS of the graph G . Thus $\gamma_{ve}(G) \leq \gamma_{ve}(H)$. We get $\gamma_{ve}(G) = \gamma_{ve}(H)$. $\square$

Observation 3. Let H be a graph with a leaf u adjacent to a weak support vertex v . Let G be a graph obtained from H by joining a path $P_4 : xyzw$ to the leaf u . Let u be adjacent to x . Then $\gamma_t(G) = \gamma_t(H) + 2$ and $\gamma_{ve}(G) = \gamma_{ve}(H) + 1$.

Proof. Let D' be a $\gamma_t(H)$ -set. It is obvious that $D' \cup \{z, y\}$ is a TDS of the graph G . Thus $\gamma_t(G) \leq \gamma_t(H) + 2$. Let D be a $\gamma_t(G)$ -set. By observation 1, $z \in D$. To dominate z , the vertex $y \in D$. It is clear that $D \setminus \{y, z\}$ is a TDS of the graph H . Thus $\gamma_t(H) \leq \gamma_t(G) - 2$. We get $\gamma_t(G) = \gamma_t(H) + 2$.

Let D' be a $\gamma_{ve}(H)$ -set. It is clear that $D' \cup \{y\}$ is a VEDS of the graph G . Thus $\gamma_{ve}(G) \leq \gamma_{ve}(H) + 1$. Let D be a $\gamma_{ve}(G)$ -set. To dominate the edge zw , the vertex $y \in D$. It is obvious that $D \setminus \{y\}$ is a VEDS of the graph H . Thus $\gamma_{ve}(H) \leq \gamma_{ve}(G) - 1$. We get $\gamma_{ve}(H) = \gamma_{ve}(G) - 1$. $\square$

First we show that if T is a nontrivial tree of order n with ℓ leaves, then $\gamma_{ve}(T)$ is bounded below by $(\gamma_t(T) - \ell + 1)/2$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 = P_5$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by one of the following operations.

- Operation $\mathcal{O}_1$: Attach a path P_2 by joining one of its vertices to a vertex of T_k adjacent to a path P_2 .

- Operation $\mathcal{O}_2$: Attach a path P_4 by joining one of its leaves to a leaf of T_k whose support vertex is weak.

Lemma 4. *If $T \in \mathcal{T}$, then $\gamma_{ve}(T) = (\gamma_t(T) - \ell + 1)/2$.*

Proof. We use the induction on the number k of operations performed to construct the tree T . If $T_1 = P_5$, then $\gamma_{ve}(T_1) = 1$ and $\gamma_t(T_1) = 3$. It can be verified that $\gamma_{ve}(T_1) = (\gamma_t(T_1) - \ell + 1)/2$ is satisfied. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the family $\mathcal{T}$ constructed by $k - 1$ operations. Let ℓ' be the number of leaves of the tree T' . Let $T = T_{k+1}$ be a tree of the family $\mathcal{T}$ constructed by k operations.

First assume that T is obtained from T' by operation $\mathcal{O}_1$. Let x be a vertex to which a path $P_2 = yz$ is attached. Let x be adjacent to y . Let uv be a path different from yz attached at x . Let u be adjacent to x . By observation 2, $\gamma_t(T) = \gamma_t(T') + 1$ and $\gamma_{ve}(T) = \gamma_{ve}(T')$. It is easy to see $\ell' = \ell - 1$. We get $(\gamma_t(T) - \ell + 1)/2 = (\gamma_t(T') + 1 - (\ell' + 1) + 1)/2 = \gamma_{ve}(T') = \gamma_{ve}(T)$.

Now assume that T is obtained from T' by operation $\mathcal{O}_2$. Let x be a leaf to which a path $P_4 : uvwz$ is attached. Let x be adjacent to u . By observation 3, $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_{ve}(T) = \gamma_{ve}(T') - 1$. It is easy to see $\ell' = \ell$. We get $(\gamma_t(T) - \ell + 1)/2 = (\gamma_t(T') + 2 - \ell' + 1)/2 = (\gamma_t(T') - \ell' + 1)/2 + 1 = \gamma_{ve}(T') + 1 = \gamma_{ve}(T)$. $\square$

Theorem 5. *If T is a tree with ℓ leaves then $\gamma_{ve}(T) \geq (\gamma_t(T) - \ell + 1)/2$ with equality if and only if $T \in \mathcal{T}$.*

Proof. If $\text{diam}(T) = 1$, then $T = P_2$. We have $(\gamma_t(T) - \ell + 1)/2 = (2 - 2 + 1)/2 < 1 = \gamma_{ve}(T)$. If $\text{diam}(T) = 2$, then T is a star $K_{1,n-1}$. We have $(\gamma_t(T) - \ell + 1)/2 = (2 - (n - 1) + 1)/2 = (4 - n)/2 < 1 = \gamma_{ve}(T)$ as $n \geq 3$. Now assume that $\text{diam}(T) \geq 3$. Thus the order n of the tree is at least four. We prove the result by induction on n . Assume that the theorem is true for every tree T' of order $n' < n$ with ℓ' leaves.

First assume that some support vertex of T , say x , is strong. Let y and z be two leaves adjacent to x . Let $T' = T - y$. Let D' be a $\gamma_t(T')$ -set. By observation 1, $x \in D'$. To dominate x , a vertex in $N_G(x)$ is in D' . Clearly the vertex x dominates y in the tree T . The set D' is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T')$. It is clear that $\ell' = \ell - 1$. Let D be a $\gamma_{ve}(T)$ -set. It is obvious that D is a VEDS of the tree T' . Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T)$. We now

get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_t(T') - \ell' - 1 + 1)/2 = \gamma_{ve}(T') - 1/2 < \gamma_{ve}(T)$. Henceforth, every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $diam(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , and let u be the parent of v in the rooted tree. If $diam(T) \geq 4$, then let w be the parent of u . If $diam(T) \geq 5$, then let d be the parent of w . If $diam(T) \geq 6$, then let e be the parent of d . By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree.

From the last but one paragraph, we get $d_T(v) = 2$. Assume that among the children of u there is a support vertex, say x , other than v . Let y be the leaf adjacent to x . Let $T' = T - T_x$. We have $\ell' = \ell - 1$. Let D' be a $\gamma_t(T')$ -set. By observation 1, $v \in D'$. To dominate v , the vertex $u \in D'$. Clearly $D' \cup \{x\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 1$. Let D be a $\gamma_{ve}(T)$ -set. To dominate the edge vt and xy , the vertex $u \in D$. Clearly D is a VEDS of the tree T' . Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T)$. We now get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_t(T') + 1 - \ell' - 1 + 1)/2 = \gamma_{ve}(T') \leq \gamma_{ve}(T)$. This implies that $(\gamma_t(T') - \ell' + 1)/2 = \gamma_{ve}(T')$. By the induction hypothesis we have $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_1$. Thus $T \in \mathcal{T}$.

Assume that among the children of u , other than v , there is a leaf x . Let $T' = T - x$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_t(T')$ -set. By observation 1, $v \in D'$. To dominate v , the vertex $u \in D'$. Clearly D' is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T')$. Let D be a $\gamma_{ve}(T)$ -set. To dominate the edge vt , the vertex $u \in D$. It is easy to see that D is a VEDS of the tree T' . Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T)$. We now get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_t(T') - \ell' - 1 + 1)/2 = \gamma_{ve}(T') - 1/2 < \gamma_{ve}(T)$.

We assume that $d_T(u) = 2$. Now assume that among the children of w , other than u , there is a vertex x such that the distance of w to the most distant vertex of T_x is three. It suffices to consider that w is adjacent to a path $P_3 : xyz$. Let $T' = T - T_u$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_t(T')$ -set. It is easy to see that $D' \cup \{u, v\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Let D be a $\gamma_{ve}(T)$ -set. To dominate the edges vt and yz , the vertices $u, x \in D$. It is easy to observe that $D \setminus \{u\}$ is a VEDS of the tree T' . Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_t(T') + 2 - \ell' - 1 + 1)/2 = \gamma_{ve}(T') + 1/2 < \gamma_{ve}(T)$.

Assume that among the children of w , other than u , there is a vertex x such that the distance of w to the most distant vertex of T_x is two. It suffices to consider that w is adjacent to a path $P_2 : xy$. Let $T' = T - T_u$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_t(T')$ -set. It is easy to observe that $D' \cup \{u, v\}$

is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Let D be a $\gamma_{ve}(T)$ -set. To dominate the edges vt and yx , the vertices $u, w \in D$. It is clear that $D \setminus \{u\}$ is a VEDS of the tree T' . Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_t(T') + 2 - \ell' - 1 + 1)/2 = \gamma_{ve}(T') + 1/2 < \gamma_{ve}(T)$.

Assume that among the children of w , other than u , there is a leaf x . Let $T' = T - T_w$. We have $\ell = \ell' + 2$. Let D' be a $\gamma_t(T')$ -set. It is easy to see that $D' \cup \{u, v, w\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 3$. Let D be a $\gamma_{ve}(T)$ -set. To dominate the edges vt, uv, uw, wx and wd , the vertex $u \in D$. It is clear that $D \setminus \{u\}$ is a VEDS of the tree T' . Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_t(T') + 3 - \ell' - 2 + 1)/2 = \gamma_{ve}(T') + 1/2 < \gamma_{ve}(T)$.

Assume $d_T(d) \geq 3$. Let $T' = T - T_w$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_t(T')$ -set. It is clear that $D' \cup \{u, v\}$ is a TDS of the tree T . Thus $\gamma_t(T) \leq \gamma_t(T') + 2$. Let D be a $\gamma_{ve}(T)$ -set. To dominate the edges vt, uv, uw and wd , the vertex $u \in D$. It is clear that $D \setminus \{u\}$ is a VEDS of the tree T' . Thus $\gamma_{ve}(T') \leq \gamma_{ve}(T) - 1$. We now get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_t(T') + 2 - \ell' - 1 + 1)/2 = \gamma_{ve}(T') + 1/2 < \gamma_{ve}(T)$.

Suppose $d = r$, we have $T = P_5 = rwuv$. Then $(\gamma_t(T) - \ell + 1)/2 = (3 - 2 + 1)/2 = 1 = \gamma_{ve}(T)$. Thus $T \in \mathcal{T}$. Now assume that $d_T(d) = 2$. Let $T' = T - T_w$. We have $\ell = \ell'$. Placing the arguments as in the previous case, we get $(\gamma_t(T) - \ell + 1)/2 \leq (\gamma_t(T') + 2 - \ell' + 1)/2 \leq \gamma_{ve}(T') + 1 \leq \gamma_{ve}(T)$. This implies that $(\gamma_t(T') - \ell' + 1)/2 = \gamma_{ve}(T')$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. $\square$

We now show that if T is a nontrivial tree of order n with ℓ leaves, then $\gamma_{ve}(T)$ is bounded above by $(\gamma_t(T) + \ell - 2)/2$. For the purpose of characterizing the trees attaining this bound we introduce a family $\mathcal{F}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \{P_2, P_3, P_4\}$. If k is a positive integer, then T_{k+1} can be obtained recursively from T_k by operation $\mathcal{O}_2$.

It is easy to see that $\mathcal{F}$ consists of paths P_n where $n \neq 4k + 1$ for positive integer k .

Lemma 6. *If $T \in \mathcal{T}$, then $\gamma_{ve}(T) = (\gamma_t(T) + \ell - 2)/2$.*

Proof. We use the induction on the number k of operations performed to construct the tree T . If $T_1 \in \{P_2, P_3, P_4\}$, then $\gamma_{ve}(T_1) = 1$ and $\gamma_t(T_1) = 2$. It can be verified that $\gamma_{ve}(T_1) = (\gamma_t(T_1) + \ell - 2)/2$ is satisfied. Let $k \geq 2$ be an integer. Assume that the result is true for every tree $T' = T_k$ of the

family $\mathcal{F}$ constructed by $k - 1$ operations. Let ℓ' be the number of leaves of the tree T' . Let $T = T_{k+1}$ be a tree of the family $\mathcal{F}$ constructed by k operations.

Assume that T is obtained from T' by operation $\mathcal{O}_2$. Let x be a leaf to which a path $P_4 : uvwz$ is attached. Let x be adjacent to u . By observation 3, $\gamma_t(T) = \gamma_t(T') + 2$ and $\gamma_{ve}(T') = \gamma_{ve}(T) - 1$. It is easy to see $\ell' = \ell$. We get $(\gamma_t(T) + \ell - 2)/2 = (\gamma_t(T') + 2 + \ell' - 2)/2 = (\gamma_t(T') + \ell' - 2)/2 + 1 = \gamma_{ve}(T') + 1 = \gamma_{ve}(T)$. $\square$

Theorem 7. *If T is a tree with ℓ leaves then $\gamma_{ve}(T) \leq (\gamma_t(T) + \ell - 2)/2$ with equality if and only if $T \in \mathcal{F}$.*

Proof. If $\text{diam}(T) = 1$, then $T = P_2$. We have $(\gamma_t(T) + \ell - 2)/2 = (2 + 2 - 2)/2 = 1 = \gamma_{ve}(T)$. Thus $T \in \mathcal{F}$. If $\text{diam}(T) = 2$. If T is a path P_3 , we have $(\gamma_t(T) + \ell - 2)/2 = (2 + 2 - 2)/2 = 1 = \gamma_{ve}(T)$. Thus $T \in \mathcal{F}$. T is a star $K_{1,n-1}$ other than P_3 . We have $(\gamma_t(T) + \ell - 2)/2 = (2 + (n - 1) - 2)/2 = (n - 1)/2 > 1 = \gamma_{ve}(T)$ as $n > 3$. Now assume that $\text{diam}(T) \geq 3$. Thus the order n of the tree is at least four. We prove the result by induction on n . Assume that the theorem is true for every tree T' of order $n' < n$ with ℓ' leaves.

First assume that some support vertex of T , say x , is strong. Let y and z be two leaves adjacent to x . Let $T' = T - y$. We have $\ell' = \ell - 1$. Let D' be a $\gamma_{ve}(T')$ -set. The vertex which dominates the edge xz in the tree T' dominates the edge xy in the tree T . It is obvious that D' is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Let D be a $\gamma_t(T)$ -set. By observation 1, $x \in D$. To dominate x , a vertex in $N_G(x)$ is in D . If $y \in D$ then $(D \setminus \{y\}) \cup \{z\}$ is a TDS of the tree T' . Thus $\gamma_t(T') \leq \gamma_t(T)$. We now get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq (\gamma_t(T') + \ell' - 2)/2 \leq (\gamma_t(T) + \ell - 1 - 2)/2 < (\gamma_t(T) + \ell - 2)/2$. Henceforth, every support vertex of T is weak.

We now root T at a vertex r of maximum eccentricity $\text{diam}(T)$. Let t be a leaf at maximum distance from r , v be the parent of t , and let u be the parent of v in the rooted tree. If $\text{diam}(T) \geq 4$, then let w be the parent of u . If $\text{diam}(T) \geq 5$, then let d be the parent of w . If $\text{diam}(T) \geq 6$, then let e be the parent of d . By T_x we denote the subtree induced by a vertex x and its descendants in the rooted tree.

Assume that among the children of u , other than v , there is a leaf x . Let $T' = T - x$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_{ve}(T')$ -set. To dominate the edge vt , the vertex $u \in D'$. Clearly D' is a VEDS of the tree T as u dominates the edge ux . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Let D be a $\gamma_t(T)$ -set. By

observation 1, the vertices $u, v \in D$. It is obvious that D is a TDS of the tree T' . Thus $\gamma_t(T') \leq \gamma_t(T)$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq (\gamma_t(T') + \ell' - 2)/2 \leq (\gamma_t(T) + \ell - 1 - 2)/2 < (\gamma_t(T) + \ell - 2)/2$.

Assume that among the children of u there is a support vertex, say x , other than v . Let y be the leaf adjacent to x . Let $T' = T - T_x$. We have $\ell' = \ell - 1$. Let D' be a $\gamma_{ve}(T')$ -set. To dominate the edge vt , the vertex $u \in D'$. The vertex u dominates the edges ux and xy in the tree T . Clearly D' is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T')$. Let D be a $\gamma_t(T)$ -set. By observation 1, the vertices $v, x \in D$. To dominate v and x , the vertex $u \in D$. It is obvious that $D \setminus \{x\}$ is a TDS of the tree T' . Thus $\gamma_t(T') \leq \gamma_t(T) - 1$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') \leq (\gamma_t(T') + \ell' - 2)/2 \leq (\gamma_t(T) - 1 + \ell - 1 - 2)/2 < (\gamma_t(T) + \ell - 2)/2$.

We assume that $d_T(u) = 2$. Now assume that among the children of w , other than u , there is a vertex x such that the distance of w to the most distant vertex of T_x is three. It suffices to consider that w is adjacent to a path $P_3 : xyz$. Let $T' = T - T_u$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_{ve}(T')$ -set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$ -set. By observation 1, the vertices $y, v \in D$. To dominate the two vertices y and v the vertices $x, u \in D$. It is clear that $D \setminus \{u, v\}$ is a TDS of the tree T' . Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(T) - 2 + \ell - 1 - 2)/2 + 1 < (\gamma_t(T) + \ell - 2)/2$.

Assume that among the children of w , other than u , there is a vertex x such that the distance of w to the most distant vertex of T_x is two. It suffices to consider that w is adjacent to a path $P_2 : xy$. Let $T' = T - T_u$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_{ve}(T')$ -set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$ -set. By observation 1, the vertices $x, v \in D$. To dominate x, v the vertices $w, u \in D$. It is clear that $D \setminus \{u, v\}$ is a TDS of the tree T' . Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(T) - 2 + \ell - 1 - 2)/2 + 1 < (\gamma_t(T) + \ell - 2)/2$.

Assume that among the children of w , other than u , there is a leaf x . Let $T' = T - T_w$. We have $\ell = \ell' + 2$. Let D' be a $\gamma_{ve}(T')$ -set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$ -set. By observation 1, the vertices $v, w \in D$. To dominate v and w , the vertex $u \in D$. It is obvious that $(D \setminus \{w, u, v\}) \cup \{a\}$ where a is a vertex in $N_G(d)$ other than w is a TDS of the tree T' . Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(T) - 2 + \ell - 2 - 2)/2 + 1 < (\gamma_t(T) + \ell - 2)/2$.

Assume $d_T(d) \geq 3$. Let $T' = T - T_w$. We have $\ell = \ell' + 1$. Let D' be a $\gamma_{ve}(T')$ -set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$ -set. By observation 1, $v \in D$. To dominate v , the vertex $u \in D$. It is clear that $D \setminus \{u, v\}$ is a TDS of the tree T' . Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(T) - 2 + \ell - 1 - 2)/2 + 1 < (\gamma_t(T) + \ell - 2)/2$.

Now assume $d_T(d) = 2$. Let $T' = T - T_w$. We have $\ell = \ell'$. Let D' be a $\gamma_{ve}(T')$ -set. It is obvious that $D' \cup \{u\}$ is a VEDS of the tree T . Thus $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1$. Let D be a $\gamma_t(T)$ -set. By observation 1, $v \in D$. To dominate v , the vertex $u \in D$. It is clear that $D \setminus \{u, v\}$ is a TDS of the tree T' . Thus $\gamma_t(T') \leq \gamma_t(T) - 2$. We get $\gamma_{ve}(T) \leq \gamma_{ve}(T') + 1 \leq (\gamma_t(T') + \ell' - 2)/2 + 1 \leq (\gamma_t(T) - 2 + \ell - 2)/2 + 1 \leq (\gamma_t(T) + \ell - 2)/2$. This implies that $\gamma_{ve}(T') = (\gamma_t(T') + \ell' - 2)/2$. By the inductive hypothesis $T' \in \mathcal{T}$. The tree T can be obtained from T' by operation $\mathcal{O}_2$. Thus $T \in \mathcal{T}$. $\square$

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