On the uniform ergodic theorem in invariant subspaces

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Abstract

Let $T$ be a bounded linear operator on a Banach space $X$ into itself.

In this paper, we study the uniform ergodicity of the operator $T|_Y$ when $Y$ is a closed subspace invariant under $T$. We show that if $T$ satisfies $\lim_{n \to \infty} \frac{\|T^n\|}{n} = 0$, then $T$ is uniformly ergodic on $X$ if and only if the restriction of $T$ to some closed subspace $Y \subset X$, invariant under $T$ and $R[(I - T)^k] \subset Y$ for some integer $k \geq 1$, is uniformly ergodic. Consequently, we obtain other equivalent conditions concerning the theorem of Mbekhta and Zemânek [9, theorem 1], also to the theorem of the Gelfand-Hille type.

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1. Introduction

Throughout this paper, $B(X)$ denotes the Banach algebra of all bounded linear operators on a Banach space $X$ into itself. For $T \in B(X)$, we denote by $R(T)$, $N(T)$, $\sigma(T)$, $r(T)$ and $\rho(T)$, the range, the kernel, the spectrum, the spectral radius and the resolvent set of $T$, respectively. A closed subspace $Y$ of $X$ is called invariant under $T$ or shortly $T$-invariant if $T(Y) \subset Y$.

By $M_n(T) = \frac{1}{n} \sum_{k=0}^{n-1} T^k$, for $n \in \mathbb{N}$, denote the sequence of the arithmetic means of the powers of $T$. An operator $T \in B(X)$ is called uniformly ergodic (resp. mean ergodic) if $\{M_n(T)\}$ is uniformly (resp. strongly) convergent in $B(X)$.

Recall that, $T \in B(X)$ is called Cesàro bounded if $\sup_n \|M_n(T)\| < \infty$. Then a every uniformly ergodic operator $T$ is necessarily Cesàro bounded, and $\frac{\|T^n\|}{n} \to 0$ when $n \to \infty$, by the following identity:

$$T^n = (n + 1)M_{n+1}(T) - M_n(T).$$

In 1938, Yosida [12] showed that when $\{M_n(T)\}$ converge strongly to $P \in B(X)$ (i.e. $T$ is mean ergodic) then $P$ is the projection onto the space $N(I - T)$ along $R(I - T)$, corresponding to the ergodic decomposition (see [7, Theorem 2.1.3.])

$$X = N(I - T) \oplus R(I - T).$$

Much of ergodic operator theory is concerned with determining when, conversely, the convergence to 0 of $\{\frac{T^n}{n}\}$ and/or the boundedness of $\{M_n(T)\}$ implies the convergence of $\{M_n(T)\}$ in the operator topology considered. A fundamental theorem due to M. Lin [8] (see also [7, Theorem 2.2.1. p.87]) says that when $\frac{\|T^n\|}{n} \to 0$, $T$ is uniformly ergodic if and only if $R(I - T)$ is closed. In this case $R[(I - T)k]$ is closed for every integer $k \geq 1$.

Mbekhta and Zemànek [9], still assuming $\frac{\|T^n\|}{n} \to 0$ when $n \to \infty$, have showed that $T$ is uniformly ergodic if and only if $R[(I - T)k]$ is closed for some integer $k \geq 1$. The case $k = 2$, is due to Dunford [4]. On the other hand, Koliha in [6, Theorem 4] proved that $\{T^n, n \geq 0\}$ converge uniformly if and only if $r(T) \leq 1$, $\sigma(T) \cap \Gamma \subset \{1\}$ and 1 is a pole of the resolvent of order at most 1 ($\Gamma$ is the unit circle). The latter condition is equivalent to uniform ergodicity (see [4]). In the same case, Mbekhta and Zemànek give another condition equivalent to Koliha’s theorem [6], presented in the following theorem:
Theorem 1.1. [9, Corollary 3] Let $T$ be a bounded linear operator on a complex Banach space $X$. Then the following conditions are equivalent:

1. $\{T^n\}_{n \geq 0}$ converge uniformly in $\mathcal{B}(X)$,
2. $T$ is uniformly ergodic and $\sigma(T) \cap \Gamma \subset \{1\}$,
3. $\lim_{n \to \infty} \frac{\|T^n\|}{n} = 0$, $\sigma(T) \cap \Gamma \subset \{1\}$ and $R((I - T)^k)$ is closed for some integer $k \geq 1$,
4. $\|T^n - T^{n+1}\| \to 0$ when $n \to \infty$ and $R((I - T)^k)$ is closed for some integer $k \geq 1$.

M. E. Becker [2] gives an example of a subspace of $X$ which is invariant under $T$ denoted by $X_1 := \{x \in X : \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} T^k x \text{ exists} \}$. By the Hahn-Banach theorem, $X_1 \subset \overline{R(I - T)}$. She proved [2, Remark 2] that if $\lim_{n \to \infty} \frac{\|T^n\|}{n} = 0$ and $R(I - T) \subset X_1$, then $T$ is uniformly ergodic if and only if $X_1$ is closed. When $T$ is power-bounded, $R(I - T) \subset X_1$.

This paper is organized as follows. In section 2, we give some definitions and fundamental properties of ergodic operator theory. In section 3, we show that for every operator $T \in \mathcal{B}(X)$ which satisfies $\lim_{n \to \infty} \frac{\|T^n\|}{n} = 0$, $T$ is uniformly ergodic if and only if there exists a closed subspace $Y \subset X$, $T$-invariant, contains $R((I - T)^k)$ for some integer $k \geq 1$ and $T|_Y$ is uniformly ergodic. This result was also obtained for an operator satisfying $\lim_{n \to \infty} \frac{T^n x}{n} = 0$ for all $x \in X$, in the weak operator topology.

2. Preliminaries

In this section, we briefly review the definitions and some basic properties which we will need in the sequel.

For $T \in \mathcal{B}(X)$, Recall that the ascent of $T$ is the smallest nonnegative integer $n$ such that $N(T^n) = N(T^{n+1})$, if no such $n$ exists, we write $\text{asc}(T) = \infty$. Similarly, the descent of $T$ is the smallest nonnegative integer $n$ such that $R(T^n) = R(T^{n+1})$, if there is no such $n$, we write $\text{des}(T) = \infty$ (see e.g. [1, Definition 3.1] or [3] p.10). It may be instructive to note that one may have $\text{des}(T) \leq n < \infty$ without $R(T^n)$ begin closed (see example at the end of [5]). We mention the following characterization:
Lemma 2.1. [6, Lemma 1.1] Given a non-negative integer $d$ and $T \in B(X)$, we have

(i) $\text{asc}(T) \leq d < \infty$ if and only if $R(T^d) \cap N(T^m) = \{0\}$, for some (equivalently, all) integer $m \geq 1$.
(ii) $\text{des}(T) \leq d < \infty$ if and only if $R(T^m) + N(T^d) = X$, for some (equivalently, all) integer $m \geq 1$.

If both $\text{asc}(T)$ and $\text{des}(T)$ are finite, then they are equal, and $X = R(T^d) \oplus N(T^d)$

where $d = \text{asc}(T) = \text{des}(T)$. For more details see Lemma 1.4.6, Lemma 3.2.4 and Proposition 1.4.3 from [3] or p.330 from [11].

Lemma 2.2. Let $T$ be a bounded linear operator on a real or complex Banach space $X$. Let $Y \subset X$ be a closed subspace which is $T$-invariant and $T|_Y$ be the restriction of $T$ to $Y$.

If $T$ is uniformly ergodic (resp. mean ergodic), then $T|_Y$ is uniformly ergodic (resp. mean ergodic).

Proof:
Let $Y \subset X$ be a closed subspace which is $T$-invariant, and denote $S = T|_Y$. Since $Y$ is $T$-invariant, it is also $T^n$-invariant for each $n \in N$, thus it is also invariant under $S_n = (T^n|_Y)$. Using that $Y$ is closed, we get that both kind of limits are inside of $Y$.

Thus, $S$ is uniformly ergodic.

Proposition 2.1. [5, Remark 1.4] Let $T$ be a bounded linear operator on a complex Banach space $X$. If $T$ satisfies either of the following assumptions:

(i) $T$ is Cesàro bounded operator, or
(ii) $\frac{T^n}{m}$ converge weakly to $0$.

Then the spectral radius of $T$ is not greater than $1$. Moreover, $N(I - T) \cap R(I - T) = \{0\}$, which in turn yields $\text{asc}(I - T) \leq 1$.

The next lemma can be considered a version of the Gelfand-Hille theorem [13].

Lemma 2.3. [9, Corollary 2] Let $T$ be a bounded linear operator on a complex Banach space $X$ such that $\sigma(T) = \{1\}$. If $T$ is uniformly ergodic, then $T = I$. 
3. Main results

The first main result of this paper is the following theorem:

**Theorem 3.1.** Let $T$ be a bounded linear operator on a real or complex Banach space $X$ such that $\lim_{n \to \infty} \frac{\|T^n\|}{n} = 0$. Let $T|_Y$ be the restriction of $T$ to a closed subspace $Y \subset X$, which is invariant under $T$, and $R(I-T) \subset Y$.

If $T|_Y$ is uniformly ergodic, then $T$ is uniformly ergodic.

**Proof:**

Let $Y$ be a closed subspace of $X$ (not trivial) which is invariant under $T$, assume that $R(I-T)$ is included in $Y$, and suppose that $T|_Y$ is uniformly ergodic.

Put $Z := \overline{R(I-T)}$. Then $Z \subset Y$, so $T|_Z$ is uniformly ergodic on $Z$. The limit is 0 on $R(I-T)$, so by continuity of the limit $\|M_n(T|_Z)\| \to 0$. Hence $(I_Z - T|_Z)Z = Z$. Then

$$R(I-T) \subset Z = (I_Z - T|_Z)Z \subset R(I-T),$$

which implies $R(I-T) = Z$, so $R(I-T)$ is closed and $T$ is uniformly ergodic by [8].

**Corollary 3.1.** Let $T$ be a bounded linear operator on a complex Banach space $X$ such that $\lim_{n \to \infty} \frac{\|T^n\|}{n} = 0$. Let $Y \subset X$ be a closed subspace $T$-invariant such that $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$. If $T|_Y$ is uniformly ergodic, then $T$ is uniformly ergodic.

**Proof:**

Let $Y \subset X$ be a closed subspace $T$-invariant such that $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$. Let’s denote $S = T|_Y$, and assume that $S$ is uniformly ergodic. By uniform ergodicity, $Y = R(I-S) \oplus N(I-S)$ with $R(I-S)$ closed [8].

We will show $R[(I-T)^k] = R(I-S)$. Since $R[(I-S)^k] \subset R[(I-T)^k]$, we need to prove only $R[(I-T)^k] \subset R[(I-S)^k]$. Since $R[(I-T)^k] \subset Y = R(I-S) \oplus N(I-S)$ by assumption, for $x \in X$ write $(I-T)^k x = (I-S)y + z$ with $y, z \in Y$ and $Tz = Sz = z$. Then

$$M_n(T)(I-T)^k x = M_n(S)(I-S)y + z,$$
and \(\|T^n\|n \to 0\) yields that \(z = 0\), so \(R[(I - T)^k] \subset R(I - S)\). But by uniform ergodicity \(I - S\) is invertible on \(R(I - S)\), which is closed, so \(R[(I - S)^k] = R(I - S)\).

Hence \(R[(I - T)^k] \subset R[(I - S)^k]\) and equality holds, so \(R[(I - T)^k] = R(I - S)\) is closed.

Hence \(T\) is uniformly ergodic by [9]. □

In the following theorem, we give a generalization of Theorem 3.1 and Corollary 3.1 where \(X\) is a real or complex Banach space and \(T \in B(X)\) satisfying \(\lim_{n \to \infty} T^n x/n = 0\), for all \(x \in X\), in the weak operator topology.

**Theorem 3.2.** Let \(T\) be a bounded linear operator on a real or complex Banach space \(X\) such that \(\lim_{n \to \infty} T^n x/n = 0\), for all \(x \in X\), in the weak operator topology. Let \(Y \subset X\) be a closed subspace \(T\)-invariant such that \(R[(I - T)^k] \subset Y\), for some integer \(k \geq 1\).

If \(T|_Y\) is uniformly ergodic, then \(T\) is uniformly ergodic.

**Proof:**

Let \(Y \subset X\) be a closed subspace \(T\)-invariant such that \(R[(I - T)^k] \subset Y\), for some integer \(k \geq 1\). Let’s denote \(S = T|_Y\), and assume that \(S\) is uniformly ergodic.

Using the technique employed in the proof of Corollary 3.1, we get \(R[(I - T)^k] = R(I - S)\). Now, we prove that if \(k > 1\), then also \(R[(I - T)^{k-1}] = R(I - S)\).

Let \(x \in X\), then \((I - T)^k x = (I - S)y = (I - T)y\) for some \(y \in Y\), and by uniform ergodicity of \(S\), we can take \(y \in R(I - S)\). Then \((I - T)[(I - T)^{k-1}x - y] = 0\), so \((I - T)^{k-1}x = y + z\) with \(y \in R(I - S)\) and \(z \in N(I - T)\).

Applying \(M_n(T)\) to both sides and using \(T^n/n \to 0\) weakly, we get \(z = 0\), which implies \(R[(I - T)^{k-1}] \subset R(I - S)\). We proceed by induction on \(R(I - T) = R(I - S)\).

Then \(M_n(T|_{R(I - T)}) = M_n(S|_{R(I - S)})\), yields that \(\lim_{n \to \infty} \|T^n/n\| = 0\) by uniform ergodicity of \(S\). Since \(R(I - T) = R(I - S)\) is closed, \(T\) is uniformly ergodic by [8].
Theorem 3.3. Let $T$ be a Cesàro bounded operator on a real or complex Banach space $X$, and $Y$ be a closed subspace of $X$ which is invariant under $T$. If there exists $k \geq 1$ such that $R[(I - T)^k] \subset Y$ and $T|_Y$ is uniformly ergodic, then $R(I - T)$ is closed and $X = R(I - T) \oplus N(I - T)$. Moreover, $T$ is uniformly ergodic.

Proof:
Let $T$ be a Cesàro bounded operator on a real or complex Banach space $X$, then by Proposition 2.1 we have $R(I - T) \cap N(I - T) = \{0\}$, hence $asc(I - T) \leq 1$.

Let $Y \subset X$ be a closed subspace which is invariant under $T$ and denote by $S = T|_Y$.

Assume that there is an integer $k \geq 1$ such that $R[(I - T)^k] \subset Y$ and $S$ is uniformly ergodic. Since $N(I - S) \subset N(I - T)$, then $R(I - T) \cap N(I - S) = \{0\}$. Since $S$ is uniformly ergodic, then $Y = R(I - S) \oplus N(I - S)$ and $R(I - S)$ is closed. By assumption we infer that $R[(I - T)^k] \subset R(I - S)$.

The uniform ergodicity of $S$ implies that $I - S$ is invertible on $R(I - S)$, so $R[(I - S)^k] = R(I - S)$. Hence

$$R[(I - T)^k] \subset R(I - S) = R[(I - S)^k] \subset R[(I - T)^k],$$

which shows that $R[(I - T)^k] = R(I - S)$ and this yields $R[(I - T)^n] = R(I - S)$ for all $n \geq k$, hence $des(I - T) < \infty$. Thus by [11, Theorem V.6.2] $X = R(I - T) \oplus N(I - T)$.

Therefore, for all $n \geq 1$, $R(I - T) = R[(I - T)^n] = R(I - S)$. Consequently,

$$M_n(T|_{R(I - T)}) = M_n(S|_{R(I - S)})$$

which implies $\frac{\|T^n\|}{n} \to 0$ by uniform ergodicity of $S$. Since $R(I - T) = R(I - S)$ is closed, then $T$ is uniformly ergodic by [8].

Corollary 3.2. Let $T$ be a Cesàro bounded operator on a real or complex Banach space $X$. $T$ is uniformly ergodic if (and only if) the restriction of $T$ to $R[(I - T)^k]$ is uniformly ergodic, for some integer $k \geq 1$. 

\[\square\]
Now, we recall the following lemma that was introduced in [10, Theorem 2.2(ii)].

**Lemma 3.1.** Let $T$ be a Cesàro bounded operator on a complex Banach space $X$ with $\sigma(T) \cap \Gamma \subset \{1\}$, where $\Gamma$ is the unit circle, then $\lim_{n \to \infty} \frac{\|T^n\|}{n} = 0$.

In accordance with the previous lemma, we infer from Theorem 1.1 and Theorem 3.3 the following result:

**Corollary 3.3.** Let $T$ be a bounded linear operator on a complex Banach space $X$.

Then the following conditions are equivalent:

1. $\{T^n\}_{n \geq 0}$ converge uniformly in $B(X)$,
2. $T$ is uniformly ergodic and $\sigma(T) \cap \Gamma \subset \{1\}$,
3. $T$ is Cesàro bounded, $\sigma(T) \cap \Gamma \subset \{1\}$ and $R[(I-T)^k]$ is closed, for some integer $k \geq 1$,
4. $T$ is Cesàro bounded, $\sigma(T) \cap \Gamma \subset \{1\}$ and $T|_Y$ is uniformly ergodic for some closed subspace $Y \subset X$ which is $T$-invariant and $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$,
5. $T$ is Cesàro bounded, $\sigma(T) \cap \Gamma \subset \{1\}$ and the restriction of $T$ to $R[(I-T)^k]$, for some integer $k \geq 1$, is uniformly ergodic.

In analogy with the Gelfand-Hille theorem, we give the following theorem:

**Theorem 3.4.** Let $T$ be a bounded linear operator on a complex Banach space $X$ such that $\sigma(T) = \{1\}$. Let $Y \subset X$ be a closed subspace and $T$-invariant such that $R[(I-T)^k] \subset Y$, for some integer $k \geq 1$. If $T|_Y$ is uniformly ergodic, then $(I-T)^{k+1} = 0$.

**Proof:**

Let $S = T|_Y$ and suppose that there is $k \geq 1$ such that $R[(I-T)^k] \subset Y$.

Assume that $S$ is uniformly ergodic, then $Y = R(I-S) \oplus N(I-S)$ and $R(I-S)$ is closed.

Since $\sigma(S) \subset \sigma(T) = \{1\}$, we have $\sigma(S) = \{1\}$. Lemma 2.3 implies $S = I$, which yields $Y = N(I-S)$. Since $R[(I-T)^k] \subset Y = N(I-S) \subset N(I-T)$, then $R[(I-T)^{k+1}] = \{0\}$, which means $(I-T)^{k+1} = 0$. 


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