On the uniform ergodic theorem in invariant subspaces

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Abstract

Let T be a bounded linear operator on a Banach space X into itself.

In this paper, we study the uniform ergodicity of the operator $T|_Y$ when Y is a closed subspace invariant under T . We show that if T satisfies $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n} = 0$, then T is uniformly ergodic on X if and only if the restriction of T to some closed subspace $Y \subset X$, invariant under T and $R[(I - T)^k] \subset Y$ for some integer $k \geq 1$, is uniformly ergodic. Consequently, we obtain other equivalent conditions concerning the theorem of Mbekhta and Zemànek [9, theorem 1], also to the theorem of the Gelfand-Hille type.

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1. Introduction

Throughout this paper, $\mathbf{B}(X)$ denotes the Banach algebra of all bounded linear operators on a Banach space X into itself. For $T \in \mathbf{B}(X)$, we denote by $R(T)$, $N(T)$, $\sigma(T)$, $r(T)$ and $\rho(T)$, the range, the kernel, the spectrum, the spectral radius and the resolvent set of T , respectively. A closed subspace Y of X is called invariant under T or shortly T -invariant if $T(Y) \subset Y$.

By $M_n(T) = \frac{1}{n} \sum_{k=0}^{n-1} T^k$, for $n \in \mathbb{N}$, denote the sequence of the arithmetic means of the powers of T . An operator $T \in \mathbf{B}(X)$ is called uniformly ergodic (resp. mean ergodic) if $\{M_n(T)\}$ is uniformly (resp. strongly) convergent in $\mathbf{B}(X)$.

Recall that, $T \in \mathbf{B}(X)$ is called Cesàro bounded if $\sup_n \|M_n(T)\| < \infty$. Then a every uniformly ergodic operator T is necessarily Cesàro bounded, and $\frac{\|T^n\|}{n} \rightarrow 0$ when $n \rightarrow \infty$, by the following identity:

$$T^n = (n+1)M_{n+1}(T) - M_n(T).$$

In 1938, Yosida [12] showed that when $\{M_n(T)\}$ converge strongly to $P \in \mathbf{B}(X)$ (i.e. T is mean ergodic) then P is the projection onto the space $N(I - T)$ along $\overline{R(I - T)}$, corresponding to the ergodic decomposition (see [7, Theorem 2.1.3.])

$$X = N(I - T) \oplus \overline{R(I - T)}.$$

Much of ergodic operator theory is concerned with determining when, conversely, the convergence to 0 of $\{\frac{T^n}{n}\}$ and/or the boundedness of $\{M_n(T)\}$ implies the convergence of $\{M_n(T)\}$ in the operator topology considered. A fundamental theorem due to M. Lin [8] (see also [7, Theorem 2.2.1. p.87]) says that when $\frac{\|T^n\|}{n} \rightarrow 0$, T is uniformly ergodic if and only if $R(I - T)$ is closed. In this case $R[(I - T)^k]$ is closed for every integer $k \geq 1$.

Mbekhta and Zemànek [9], still assuming $\frac{\|T^n\|}{n} \rightarrow 0$ when $n \rightarrow \infty$, have showed that T is uniformly ergodic if and only if $R[(I - T)^k]$ is closed for some integer $k \geq 1$. The case $k = 2$, is due to Dunford [4]. On the other hand, Koliha in [6, Theorem 4] proved that $\{T^n, n \geq 0\}$ converge uniformly if and only if $r(T) \leq 1$, $\sigma(T) \cap \Gamma \subset \{1\}$ and 1 is a pole of the resolvent of order at most 1 (Γ is the unit circle). The latter condition is equivalent to uniform ergodicity (see [4]). In the same case, Mbekhta and Zemànek give another condition equivalent to Koliha's theorem [6], presented in the following theorem:

Theorem 1.1. [9, Corollary 3] Let T be a bounded linear operator on a complex Banach space X . Then the following conditions are equivalent:

1. $\{T^n\}_{n \geq 0}$ converge uniformly in $\mathbf{B}(X)$,
2. T is uniformly ergodic and $\sigma(T) \cap \Gamma \subset \{1\}$,
3. $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n} = 0$, $\sigma(T) \cap \Gamma \subset \{1\}$ and $R[(I - T)^k]$ is closed for some integer $k \geq 1$,
4. $\|T^n - T^{n+1}\| \rightarrow 0$ when $n \rightarrow \infty$ and $R[(I - T)^k]$ is closed for some integer $k \geq 1$.

M. E. Becker [2] gives an example of a subspace of X which is invariant under T denoted by $X_1 := \{x \in X : \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{T^k x}{k} \text{ exists}\}$. By the Hahn-Banach theorem, $X_1 \subset \overline{R(I - T)}$. She proved [2, Remark 2] that if $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n} = 0$ and $R(I - T) \subset X_1$, then T is uniformly ergodic if and only if X_1 is closed. When T is power-bounded, $R(I - T) \subset X_1$.

This paper is organized as follows. In section 2, we give some definitions and fundamental properties of ergodic operator theory. In section 3, we show that for every operator $T \in \mathbf{B}(X)$ which satisfies $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n} = 0$, T is uniformly ergodic if and only if there exists a closed subspace $Y \subset X$, T -invariant, contains $R[(I - T)^k]$ for some integer $k \geq 1$ and $T|_Y$ is uniformly ergodic. This result was also obtained for an operator satisfying $\lim_{n \rightarrow \infty} \frac{T^n x}{n} = 0$ for all $x \in X$, in the weak operator topology.

2. Preliminaries

In this section, we briefly review the definitions and some basic properties which we will need in the sequel.

For $T \in \mathbf{B}(X)$, Recall that the ascent of T is the smallest non negative integer n such that $N(T^n) = N(T^{n+1})$, if no such n exists, we write $asc(T) = \infty$. Similarly, the descent of T is the smallest nonnegative integer n such that $R(T^n) = R(T^{n+1})$, if there is no such n , we write $des(T) = \infty$ (see e.g. [1, Definition 3.1] or [3] p.10). It may be instructive to note that one may have $des(T) \leq n < \infty$ without $R(T^n)$ being closed (see example at the end of [5]). We mention the following characterization:

Lemma 2.1. [6, Lemma 1.1] Given a non-negative integer d and $T \in \mathbf{B}(X)$, we have

(i) $\text{asc}(T) \leq d < \infty$ if and only if $R(T^d) \cap N(T^m) = \{0\}$, for some (equivalently, all) integer $m \geq 1$.

(ii) $\text{des}(T) \leq d < \infty$ if and only if $R(T^m) + N(T^d) = X$, for some (equivalently, all) integer $m \geq 1$.

If both $\text{asc}(T)$ and $\text{des}(T)$ are finite, then they are equal, and $X = R(T^d) \oplus N(T^d)$

where $d = \text{asc}(T) = \text{des}(T)$. For more details see Lemma 1.4.6, Lemma 3.2.4 and Proposition 1.4.3 from [3] or p.330 from [11].

Lemma 2.2. Let T be a bounded linear operator on a real or complex Banach space X . Let $Y \subset X$ be a closed subspace which is T -invariant and $T|_Y$ be the restriction of T to Y .

If T is uniformly ergodic (resp. mean ergodic), then $T|_Y$ is uniformly ergodic (resp. mean ergodic).

Proof:

Let $Y \subset X$ be a closed subspace which is T -invariant, and denote $S = T|_Y$. Since Y is T -invariant, it is also T^n -invariant for each $n \in \mathbb{N}$, thus it is also invariant under $S_n = (T^n|_Y)$. Using that Y is closed, we get that both kind of limits are inside of Y .

Thus, S is uniformly ergodic. □

Proposition 2.1. [5, Remark 1.4] Let T be a bounded linear operator on a complex Banach space X . If T satisfies either of the following assumptions:

- (i) T is Cesàro bounded operator, or
- (ii) $\frac{T^n}{n}$ converge weakly to 0.

Then the spectral radius of T is not greater than 1. Moreover, $N(I - T) \cap R(I - T) = \{0\}$, which in turn yields $\text{asc}(I - T) \leq 1$.

The next lemma can be considered a version of the Gelfand-Hille theorem [13].

Lemma 2.3. [9, Corollary 2] Let T be a bounded linear operator on a complex Banach space X such that $\sigma(T) = \{1\}$. If T is uniformly ergodic, then $T = I$.

3. Main results

The first main result of this paper is the following theorem:

Theorem 3.1. *Let T be a bounded linear operator on a real or complex Banach space X such that $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n} = 0$. Let $T|_Y$ be the restriction of T to a closed subspace $Y \subset X$, which is invariant under T , and $R(I - T) \subset Y$.*

If $T|_Y$ is uniformly ergodic, then T is uniformly ergodic.

Proof:

Let Y be a closed subspace of X (not trivial) which is invariant under T , assume that $R(I - T)$ is included in Y , and suppose that $T|_Y$ is uniformly ergodic.

Put $Z := \overline{R(I - T)}$. Then $Z \subset Y$, so $T|_Z$ is uniformly ergodic on Z . The limit is 0 on $R(I - T)$, so by continuity of the limit $\|M_n(T|_Z)\| \rightarrow 0$. Hence $(I_Z - T|_Z)Z = Z$. Then

$$R(I - T) \subset Z = (I_Z - T|_Z)Z \subset R(I - T),$$

which implies $R(I - T) = Z$, so $R(I - T)$ is closed and T is uniformly ergodic by [8]. $\square$

Corollary 3.1. *Let T be a bounded linear operator on a complex Banach space X such that $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n} = 0$. Let $Y \subset X$ be a closed subspace T -invariant such that $R[(I - T)^k] \subset Y$, for some integer $k \geq 1$. If $T|_Y$ is uniformly ergodic, then T is uniformly ergodic.*

Proof:

Let $Y \subset X$ be a closed subspace T -invariant such that $R[(I - T)^k] \subset Y$, for some integer $k \geq 1$. Let's denote $S = T|_Y$, and assume that S is uniformly ergodic. By uniform ergodicity, $Y = R(I - S) \oplus N(I - S)$ with $R(I - S)$ closed [8].

We will show $R[(I - T)^k] = R(I - S)$. Since $R[(I - S)^k] \subset R[(I - T)^k]$, we need to prove only $R[(I - T)^k] \subset R[(I - S)^k]$. Since $R[(I - T)^k] \subset Y = R(I - S) \oplus N(I - S)$ by assumption, for $x \in X$ write $(I - T)^k x = (I - S)y + z$ with $y, z \in Y$ and $Tz = Sz = z$. Then

$$M_n(T)(I - T)^k x = M_n(S)(I - S)y + z,$$

and $\|T^n\|/n \rightarrow 0$ yields that $z = 0$, so $R[(I - T)^k] \subset R(I - S)$. But by uniform ergodicity $I - S$ is invertible on $R(I - S)$, which is closed, so $R[(I - S)^k] = R(I - S)$.

Hence $R[(I - T)^k] \subset R[(I - S)^k]$ and equality holds, so $R[(I - T)^k] = R(I - S)$ is closed.

Hence T is uniformly ergodic by [9]. $\square$

In the following theorem, we give a generalization of Theorem 3.1 and Corollary 3.1 where X is a real or complex Banach space and $T \in \mathbf{B}(X)$ satisfying $\lim_{n \rightarrow \infty} \frac{T^n x}{n} = 0$, for all $x \in X$, in the weak operator topology.

Theorem 3.2. *Let T be a bounded linear operator on a real or complex Banach space X such that $\lim_{n \rightarrow \infty} \frac{T^n x}{n} = 0$, for all $x \in X$, in the weak operator topology. Let $Y \subset X$ be a closed subspace T -invariant such that $R[(I - T)^k] \subset Y$, for some integer $k \geq 1$.*

If $T|_Y$ is uniformly ergodic, then T is uniformly ergodic.

Proof:

Let $Y \subset X$ be a closed subspace T -invariant such that $R[(I - T)^k] \subset Y$, for some integer $k \geq 1$. Let's denote $S = T|_Y$, and assume that S is uniformly ergodic.

Using the technique employed in the proof of Corollary 3.1, we get $R[(I - T)^k] = R(I - S)$. Now, we prove that if $k > 1$, then also $R[(I - T)^{k-1}] = R(I - S)$.

Let $x \in X$, then $(I - T)^k x = (I - S)y = (I - T)y$ for some $y \in Y$, and by uniform ergodicity of S , we can take $y \in R(I - S)$. Then $(I - T)[(I - T)^{k-1}x - y] = 0$, so $(I - T)^{k-1}x = y + z$ with $y \in R(I - S)$ and $z \in N(I - T)$.

Applying $M_n(T)$ to both sides and using $T^n/n \rightarrow 0$ weakly, we get $z = 0$, which implies $R[(I - T)^{k-1}] \subset R(I - S)$. We proceed by induction $R(I - T) = R(I - S)$.

Then $M_n(T|_{R(I-T)}) = M_n(S|_{R(I-S)})$, yields that $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n} = 0$ by uniform ergodicity of S .

Since $R(I - T) = R(I - S)$ is closed, T is uniformly ergodic by [8].

□

Theorem 3.3. *Let T be a Cesàro bounded operator on a real or complex Banach space X ,*

and Y be a closed subspace of X which is invariant under T .

If there exists $k \geq 1$ such that $R[(I - T)^k] \subset Y$ and $T|_Y$ is uniformly ergodic, then $R(I - T)$ is closed and $X = R(I - T) \oplus N(I - T)$. Moreover, T is uniformly ergodic.

Proof:

Let T be a Cesàro bounded operator on a real or complex Banach space X , then by Proposition 2.1 we have $R(I - T) \cap N(I - T) = \{0\}$, hence $\text{asc}(I - T) \leq 1$.

Let $Y \subset X$ be a closed subspace which is invariant under T and denote by $S = T|_Y$.

Assume that there is an integer $k \geq 1$ such that $R[(I - T)^k] \subset Y$ and S is uniformly ergodic. Since $N(I - S) \subset N(I - T)$, then $R(I - T) \cap N(I - S) = \{0\}$. Since S is uniformly ergodic, then $Y = R(I - S) \oplus N(I - S)$ and $R(I - S)$ is closed. By assumption we infer that $R[(I - T)^k] \subset R(I - S)$.

The uniform ergodicity of S implies that $I - S$ is invertible on $R(I - S)$, so $R[(I - S)^k] = R(I - S)$. Hence

$$R[(I - T)^k] \subset R(I - S) = R[(I - S)^k] \subset R[(I - T)^k],$$

which shows that $R[(I - T)^k] = R(I - S)$ and this yields $R[(I - T)^n] = R(I - S)$ for all $n \geq k$, hence $\text{des}(I - T) < \infty$. Thus by [11, Theorem V.6.2] $X = R(I - T) \oplus N(I - T)$.

Therefore, for all $n \geq 1$, $R(I - T) = R[(I - T)^n] = R(I - S)$. Consequently,

$$M_n(T|_{R(I - T)}) = M_n(S|_{R(I - S)})$$

which implies $\frac{\|T^n\|}{n} \rightarrow 0$ by uniform ergodicity of S . Since $R(I - T) = R(I - S)$ is closed, then T is uniformly ergodic by [8]. □

Corollary 3.2. *Let T be a Cesàro bounded operator on a real or complex Banach space X . T is uniformly ergodic if (and only if) the restriction of T to $\overline{R[(I - T)^k]}$ is uniformly ergodic, for some integer $k \geq 1$.*

Now, we recall the following lemma that was introduced in [10, Theorem 2.2(ii)].

Lemma 3.1. *Let T be a Cesàro bounded operator on a complex Banach space X with $\sigma(T) \cap \Gamma \subset \{1\}$, where Γ is the unit circle, then $\lim_{n \rightarrow \infty} \frac{\|T^n\|}{n} = 0$.*

In accordance with the previous lemma, we infer from Theorem 1.1 and Theorem 3.3 the following result:

Corollary 3.3. *Let T be a bounded linear operator on a complex Banach space X .*

Then the following conditions are equivalent:

1. $\{T^n\}_{n \geq 0}$ converge uniformly in $\mathbf{B}(X)$,
2. T is uniformly ergodic and $\sigma(T) \cap \Gamma \subset \{1\}$,
3. T is Cesàro bounded, $\sigma(T) \cap \Gamma \subset \{1\}$ and $R[(I - T)^k]$ is closed, for some integer $k \geq 1$,
4. T is Cesàro bounded, $\sigma(T) \cap \Gamma \subset \{1\}$ and $T|_Y$ is uniformly ergodic for some closed subspace $Y \subset X$ which is T -invariant and $R[(I - T)^k] \subset Y$, for some integer $k \geq 1$,
5. T is Cesàro bounded, $\sigma(T) \cap \Gamma \subset \{1\}$ and the restriction of T to $\overline{R[(I - T)^k]}$, for some integer $k \geq 1$, is uniformly ergodic.

In analogy with the Gelfand-Hille theorem, we give the following theorem:

Theorem 3.4. *Let T be a bounded linear operator on a complex Banach space X such that $\sigma(T) = \{1\}$. Let $Y \subset X$ be a closed subspace and T -invariant such that $R[(I - T)^k] \subset Y$, for some integer $k \geq 1$. If $T|_Y$ is uniformly ergodic, then $(I - T)^{k+1} = 0$.*

Proof:

Let $S = T|_Y$ and suppose that there is $k \geq 1$ such that $R[(I - T)^k] \subset Y$.

Assume that S is uniformly ergodic, then $Y = R(I - S) \oplus N(I - S)$ and $R(I - S)$ is closed.

Since $\sigma(S) \subset \sigma(T) = \{1\}$, we have $\sigma(S) = \{1\}$. Lemma 2.3 implies $S = I$, which yields $Y = N(I - S)$. Since $R[(I - T)^k] \subset Y = N(I - S) \subset N(I - T)$, then $R[(I - T)^{k+1}] = \{0\}$, which means $(I - T)^{k+1} = 0$.

□

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