Upper triangular operator matrices and limit points of the essential spectrum

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Abstract:

In this paper, we investigate the limit points set of essential spectrum of upper triangular operator matrices\
\[ M_ε \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \]

We prove that accσ_ε(\( M_ε \)) ∪ \( W = accσ_ε(A) ∪ accσ_ε(B) \) where \( W \) is the union of certain holes in accσ_ε(\( M_ε \)), which happen to be subsets of accσ_ε(\( M_ε \)) ∩ accσ_ε(A). Also, several sufficient conditions for accσ_ε(\( M_ε \)) = accσ_ε(A) ∪ accσ_ε(B) holds are given.

Keywords: Fredholm operator; Essential spectra; Limit point; Operator matrices.


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1. Introduction and Preliminaries

Let $X, Y$ be infinite dimensional complex Banach spaces and $\mathcal{B}(X, Y)$ denote the complex algebra of all bounded linear operators from $X$ to $Y$. For $Y = X$ we write $\mathcal{B}(X, X) = \mathcal{B}(X)$. If $T \in \mathcal{B}(X)$, we denote by $T^*, N(T)$, $R(T)$, $\sigma_{ap}(T)$, $\sigma_{su}(T)$, $\sigma(T)$, respectively the adjoint, the null space, the range, the approximate point spectrum, the surjectivity spectrum and the spectrum of $T$.

A bounded linear operator is called an upper semi-Fredholm (resp, lower semi-Fredholm) if $\alpha(T) = \dim N(T) < \infty$ and $R(T)$ is closed (resp, $\beta(T) = \text{codim} R(T) < \infty$). $T$ is semi-Fredholm if it is a lower or upper semi-Fredholm operator. The index of a semi Fredholm operator $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$.

$T$ is a Fredholm operator if is a lower and upper semi-Fredholm operator. The essential spectrum of $T$ is the subset of $\mathbb{C}$ defined by:

$$\sigma_e(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a Fredholm operator} \}$$

Let $T \in \mathcal{B}(X, Y)$, $T$ is said to be left Atkinson if $T$ is upper semi-Fredholm and $R(T)$ is complemented in $X$, and it is said to be right Atkinson if $T$ is lower semi-Fredholm and $N(T)$ is complemented in $X$ (see [1]). The left and right Atkinson spectra are the subsets of $\mathbb{C}$ defined respectively by:

$$\sigma_{le}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a left Atkinson operator} \}$$

$$\sigma_{re}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not a right Atkinson operator} \}$$

$\sigma_e(T)$, $\sigma_{re}(T)$ and $\sigma_{le}(T)$ are compact subset and we have

$$\sigma_e(T) = \sigma_{re}(T) \cup \sigma_{le}(T)$$

For $A \in \mathcal{B}(X)$, $B \in \mathcal{B}(Y)$, we denote by $M_C \in \mathcal{B}(X \oplus Y)$ the operator matrix acting on the product of Banach space $X \oplus Y$ [5]:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

It is well know that, in the case of infinite dimensional, the inclusion $\sigma(M_C) \subset \sigma(A) \cup \sigma(B)$, may be strict. This motivates serval authors to study the defect $(\sigma_s(A) \cup \sigma_s(B)) \setminus \sigma_s(M_C)$ where $\sigma_s$ runs different type of spectra.
If $H$ and $K$ are Hilbert spaces, Du and Pan [5] have studied the description of $\bigcap_{C \in \mathcal{B}(K,H)} \sigma(M_C)$ by showing that

$$\bigcap_{C \in \mathcal{B}(K,H)} \sigma(M_C) = \sigma_{ap}(A) \cup \sigma_{sa}(B) \cup \{\lambda \in C : \alpha(B - \lambda) \neq \beta(A - \lambda)\}$$

Han H.Y. Lee and W. Y. Lee [6] extended the result to the Banach spaces. In [3], D.S. Djordjevic give a description of $\sigma_{e}(M_C)$, he showed the following theorem.

**Theorem 1.1 (3).** For given $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ the following holds:

$$\bigcap_{C \in \mathcal{B}(Y, X)} \sigma_{e}(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup W(A, B)$$

Where $W(A, B) = \{\lambda \in C, N(B - \lambda) \text{ and } X/\overline{R(A - \lambda)}\text{are not isomorphic up to a finite dimensional subspace}\}$

In [9], the authors showed the following theorem.

**Theorem 1.2 (9).** Let $(A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$ and $C \in \mathcal{B}(Y, X)$. Then

$$\sigma_{e}(M_C) \cup W_{e} = \sigma_{e}(A) \cup \sigma_{e}(B)$$

where $W_{e}$ is the union of certain holes in $\sigma_{e}(M_C)$, which happen to be subsets of $\sigma_{e}(A) \cap \sigma_{e}(B)$.

For a compact subset $K$ of $C$, let $\text{acc}K$, $\text{int}K$, $\text{iso}K$, $\partial K$ and $\eta(K)$ be the set of all points of accumulation of $K$, the interior of $K$, the isolated points of $K$, the boundary of $K$ and the polynomially convex hull of $K$ respectively.

In this paper, we investigate the relationship between $\text{acc}\sigma_{e}(M_C)$ and $\text{acc}\sigma_{e}(A) \cup \text{acc}\sigma_{e}(B)$. We show that the passage from $\text{acc}\sigma_{e}(M_0)$ to $\text{acc}\sigma_{e}(M_C)$ can be described as follows:

$$\text{acc}\sigma_{e}(M_C) \cup W = \text{acc}\sigma_{e}(M_0) = \text{acc}\sigma_{e}(A) \cup \text{acc}\sigma_{e}(B)$$

where $W$ is the union of certain holes in $\text{acc}\sigma_{e}(M_C)$, which happen to be subsets of $\text{acc}\sigma_{e}(A) \cap \text{acc}\sigma_{e}(B)$. 
2. Main results

We start this section by proving that the limit point of essential spectrum set of a direct sum is the limit point of essential spectra of its summands.

Proposition 2.1. Let \((A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)\) and \(C \in \mathcal{B}(Y, X)\). Then

\[
\text{acc}_{\text{e}}(M_0) = \text{acc}_{\text{e}}(A) \cup \text{acc}_{\text{e}}(B)
\]

Proof. We have \(\lambda \in \text{acc}_{\text{e}}(M_0)\) if and only if \(\lambda \in \text{acc}(\sigma_{\text{e}}(A) \cup \sigma_{\text{e}}(B)) = \text{acc}(\sigma_{\text{e}}(A)) \cup \text{acc}(\sigma_{\text{e}}(B))\).

Lemma 2.1. Let \((A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)\) and \(C \in \mathcal{B}(Y, X)\). Then

\[
\text{acc}_{\text{e}}(M_C) \subseteq \text{acc}_{\text{e}}(M_0) = \text{acc}_{\text{e}}(A) \cup \text{acc}_{\text{e}}(B)
\]

Proof. Without loss of generality, let \(\lambda = 0 \notin \text{acc}_{\text{e}}(A) \cup \text{acc}_{\text{e}}(B)\), then there exists \(\varepsilon > 0\) such that for any \(\lambda, 0 < |\lambda| < \varepsilon\), we have \(A - \lambda I\) and \(B - \lambda I\) are Fredholm. According to [4, Lemma 2.1], we have \(M_C - \lambda I\) is Fredholm for any \(\lambda, 0 < |\lambda| < \varepsilon\), thus \(0 \notin \text{acc}(\sigma_{\text{e}}(M_C))\). Therefore \(\text{acc}_{\text{e}}(M_C) \subseteq \text{acc}_{\text{e}}(A) \cup \text{acc}_{\text{e}}(B)\).

Definition 2.1. Let \(T \in \mathcal{B}(X)\). We said that \(T\) has the property \(aE\) at \(\lambda \in \mathbb{C}\) if \(\lambda \notin \text{acc}_{\text{e}}(T)\).

The following lemma will be needed in the sequel.

Lemma 2.2. If two of \(M_C\), \(A\) and \(B\) have the property \(aE\) at 0, then the third is also has the property \(aE\).

Proof. i) If \(A\) and \(B\) have the property \(aE\), by lemma 2.1 \(M_C\) has the property \(aE\).

ii) If \(M_C\) and \(A\) have the property \(aE\), then \(0 \notin \text{acc}(\sigma_{\text{e}}(M_C))\) and \(0 \notin \text{acc}(\sigma_{\text{e}}(A))\), thus there exists \(\varepsilon > 0\) such that \(M_C - \lambda I\) and \(A - \lambda I\) are Fredholm for every \(\lambda, 0 < |\lambda| < \varepsilon\). From [6, Corollary 5], \(B - \lambda I\) is Fredholm for every \(\lambda, 0 < |\lambda| < \varepsilon\).

iii) If \(B\) and \(M_C\) have the property \(aE\), the proof is similar to ii).

The first main result of this paper is the following theorem.
Theorem 2.1. Let \((A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y)\) and \(C \in \mathcal{B}(Y, X)\). Then
\[
\text{acc}_e(M_C) \cup W = \text{acc}_e(A) \cup \text{acc}_e(B)
\]
where \(W\) is the union of certain holes in \(\text{acc}_e(M_C)\), which happen to be subsets of \(\text{acc}_e(B) \cap \text{acc}_e(A)\).

Proof. We first claim that, for every \(C \in \mathcal{B}(Y, X)\) we have

\[
(\text{acc}_e(A) \cup \text{acc}_e(B)) \setminus \text{acc}_e(A) \cap \text{acc}_e(B) \subseteq \text{acc}_e(M_C) \tag{1}
\]

Indeed, let \(\lambda \in (\text{acc}_e(A) \cup \text{acc}_e(B)) \setminus \text{acc}_e(A) \cap \text{acc}_e(B)\), then \(\lambda \in \text{acc}_e(A) \setminus \text{acc}_e(B)\) or \(\lambda \in \text{acc}_e(B) \setminus \text{acc}_e(A)\).

\(i)\) If \(\lambda \in \text{acc}_e(A) \setminus \text{acc}_e(B)\), then \(A\) has not the property \(aE\) at \(\lambda\) and \(B\) has the property \(aE\) at \(\lambda\). Suppose that \(\lambda \notin \text{acc}_e(M_C)\), hence \(M_C\) has the property \(aE\) at \(\lambda\), by lemma 2.2 \(A\) has the property \(aE\) at \(\lambda\), contradiction. So \(\lambda \in \text{acc}_e(M_C)\).

\(ii)\) If \(\lambda \in \text{acc}_e(B) \setminus \text{acc}_e(A)\), by the same argument of \(i)\) we have \(\lambda \in \text{acc}_e(M_C)\).

Next, we claim that for every \(C \in \mathcal{B}(Y, X)\) we have

\[
\eta(\text{acc}_e(M_C)) = \eta(\text{acc}_e(A) \cup \text{acc}_e(B)) \tag{2}
\]

Since \(\text{acc}_e(M_C) \subseteq \text{acc}_e(A) \cup \text{acc}_e(B)\), we need to prove \(\partial(\text{acc}_e(A) \cup \text{acc}_e(B)) \subseteq \partial\text{acc}_e(M_C)\). But since \(\text{int}(\text{acc}_e(M_C)) \subseteq \text{int}(\text{acc}_e(A) \cup \text{acc}_e(B))\), by the maximum modules theorem, it suffices to show that \(\partial(\text{acc}_e(A) \cup \text{acc}_e(B)) \subseteq \text{acc}_e(M_C)\). Without loss of generality, suppose \(0 \in \partial(\text{acc}_e(A) \cup \text{acc}_e(B))\). There are two cases to consider.

Case 1: If \(0 \in \text{acc}(\partial(\text{acc}_e(A) \cup \text{acc}_e(B)))\), then there exists \((\lambda_n) \subseteq \partial(\text{acc}_e(A) \cup \text{acc}_e(B))\) such that \(\lim_{n \to \infty} \lambda_n = 0\), since

\[
\partial(\text{acc}_e(A)) \subseteq \partial(\sigma_e(A)) \subseteq \sigma_e(A) \subseteq \sigma_e(M_C)
\]

and

\[
\partial(\text{acc}_e(B)) \subseteq \partial(\sigma_e(B)) \subseteq \sigma_e(B) \subseteq \sigma_e(M_C)
\]
we have, \( \lambda_n \in \sigma_e(M_C), n = 1, 2, \ldots \), hence \( 0 \in acc_e(M_C) \).

**Case 2:** If \( 0 \in iso(\partial(acc_e(A) \cup acc_e(B))) \), since \( acc_e(A) \cup acc_e(B) \) is closed, then \( iso(\partial(acc_e(A) \cup acc_e(B))) = iso(acc_e(A) \cup acc_e(B)) \).

\( 0 \in iso(acc_e(A) \cup acc_e(B)) \), thus there exists \( \varepsilon > 0 \) such that \( \lambda \notin acc(acc_e(A) \cup acc_e(B)) \) for every \( \lambda, 0 < |\lambda| < \varepsilon \). Since \( 0 \in acc_e(A) \cup acc_e(B) = acc_e(A) \cup \sigma_e(B) \), there exists \( (\mu_n) \subseteq \sigma_e(A) \cup \sigma_e(B) \) such that \( \lim_{n \to \infty} \mu_n = 0, \mu_n \neq 0 \) for all \( n \), thus there exists certain positive integer \( N \) such that \( 0 < |\mu_n| < \varepsilon \) for any \( n \geq N \). Let \( \lambda_n = \mu_{N+1+n} \), then \( \lambda_n \in iso(\sigma_e(A) \cup \sigma_e(B)) \), \( n = 1, 2, \ldots \) and \( \lim_{n \to \infty} \lambda_n = 0 \). Since \( \sigma_e(A) \) and \( \sigma_e(B) \) are closed, then

\[
iso(\sigma_e(A) \cup \sigma_e(B)) \subseteq iso(\sigma_e(A)) \cup iso(\sigma_e(B)) \\
\subseteq \partial \sigma_e(A) \cup \partial \sigma_e(B) \\
\subseteq \sigma_{le}(A) \cup \sigma_{re}(B) \subseteq \sigma_e(M_C)
\]

Then, \( \lambda_n \in iso(\sigma_e(A) \cup \sigma_e(B)) \subseteq \sigma_e(M_C), n = 1, 2, \ldots \). Since \( \lim_{n \to \infty} \lambda_n = 0 \), so \( 0 \in acc_e(M_C) \).

Therefore \( \partial(acc_e(A) \cup acc_e(B)) \subseteq acc_e(M_C) \). This proves (2).

\( acc_e(M_C) \subseteq acc_e(A) \cup acc_e(B) \) and (2) says that the passage from \( acc_e(A) \) to \( acc_e(A) \cup acc_e(B) \) is the filling in certain of the holes in \( acc_e(M_C) \). But since \( (acc_e(A) \cup acc_e(B)) \setminus acc_e(M_C) \) is contained in \( acc_e(A) \cap acc_e(B) \), it follows that the filling in certain of the holes in \( acc_e(M_C) \) should occur in \( acc_e(A) \cap acc_e(B) \).

**Corollary 2.1.** Let \( (A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y) \). If \( acc_e(A) \cap acc_e(B) \) has no interior points, then for every \( C \in \mathcal{B}(Y, X) \) we have

\[
acc_e(M_C) = acc_e(A) \cup acc_e(B)
\]

Second main result is the following theorem.

**Theorem 2.2.** Let \( (A, B) \in \mathcal{B}(X) \times \mathcal{B}(Y) \) and \( C \in \mathcal{B}(Y, X) \). Then the following assertions are equivalent

1. \( \sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B) \),
2. \( acc_e(M_C) = acc_e(A) \cup acc_e(B) \).
Proof. First we show that $W_e \subseteq W$.
Indeed, if $\lambda \in W_e$, from theorem 1.2, we have $\lambda \in (\sigma_e(A) \cup \sigma_e(B)) \setminus \sigma_e(M_C)$, then $\lambda \notin \sigma_e(M_C)$, hence $\lambda \notin acc\sigma_e(M_C)$. It suffice to show that

$$\lambda \in acc\sigma_e(A) \cup acc\sigma_e(B) = acc(\sigma_e(A) \cup \sigma_e(B))$$

Suppose that $\lambda \notin acc(\sigma_e(A) \cup \sigma_e(B))$, since $\lambda \in \sigma_e(A) \cup \sigma_e(B)$, then

$$\lambda \in \sigma_e(A) \cup \sigma_e(B) \subseteq \sigma_e(A) \cup \sigma_e(B)$$

Hence $\lambda \in \sigma_e(M_C)$, contradiction. Therefore

$$\lambda \in (acc\sigma_e(A) \cup acc\sigma_e(B)) \setminus acc\sigma_e(M_C)$$

By theorem 2.1, we have $\lambda \in W$. So $W_e \subseteq W$.

Furthermore, $W_e \subseteq W$ implies that

$$acc\sigma_e(M_C) = acc\sigma_e(A) \cup acc\sigma_e(B) \implies \sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$$

Conversely, if $\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$ let $\lambda \notin acc\sigma_e(M_C)$, without loss of generality, we assume that $0 \notin acc\sigma_e(M_C)$, then there exists $\varepsilon > 0$ such that $M_C - \lambda$ is Fredholm for all $\lambda$, $0 < |\lambda| < \varepsilon$, hence $\lambda \notin \sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$. Thus both $A - \lambda$ and $B - \lambda$ are Fredholm for every $\lambda$, $0 < |\lambda| < \varepsilon$. Therefore $0 \notin acc(\sigma_e(A)) \cup acc(\sigma_e(B))$. Since $acc\sigma_e(M_C) \subseteq acc\sigma_e(A) \cup acc\sigma_e(B)$ always holds, then $acc\sigma_e(M_C) = acc\sigma_e(A) \cup acc\sigma_e(B)$.

It is immediate to check the following result.

**Corollary 2.2.** Let $(A,B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$. If $acc\sigma_e(A) \cap acc\sigma_e(B)$ has no interior points, then for every $C \in \mathcal{B}(Y,X)$, we have we have

$$\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B) \quad (**)$$

In particular, if either $A \in \mathcal{B}(X)$ or $B \in \mathcal{B}(Y)$ is a Riesz, then (**) holds.

Now, For $(A,B) \in \mathcal{B}(X) \times \mathcal{B}(Y)$, let $L_A$ (resp $R_B$) be the left (resp.

right) multiplication operator given by $L_A(X) = AX$; (resp. $R_B(X) = XB$), and let $\delta_{A,B}(X) = AX - XB = L_A(X) - R_B(X)$ be the usual generalized derivation associated with $A$ and $B$. We denote by $N^\infty(A) = \bigcup_{n \geq 1} N(A^n)$ the generalized kernel of $A$ [1].
Corollary 2.3. Let $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$. If $C$ is in the closure of the set
\[ R(\delta_{A,B}) + N(\delta_{A,B}) + \bigcup_{\lambda \in \mathcal{C}} N^\infty(L_{A-\lambda}) + \bigcup_{\lambda \in \mathcal{C}} N^\infty(R_{B-\lambda}) \]
then:
\[ \text{acc}\sigma_e(M_C) = \text{acc}\sigma_e(A) \cup \text{acc}\sigma_e(B) \]

Proof. If $C$ is in the closure of the set
\[ R(\delta_{A,B}) + N(\delta_{A,B}) + \bigcup_{\lambda \in \mathcal{C}} N^\infty(L_{A-\lambda}) + \bigcup_{\lambda \in \mathcal{C}} N^\infty(R_{B-\lambda}) \]
then $\sigma_e(M_C) = \sigma_e(A) \cup \sigma_e(B)$, by [2]. From theorem 2.2, we obtain the result.

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