Equitable total chromatic number of splitting graph

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Abstract:

Among the various coloring of graphs, the concept of equitable total coloring of graph \( G \) is the coloring of all its vertices and edges in which the number of elements in any two color classes differ by at most one. The minimum number of colors required is called its equitable total chromatic number. In this paper, we determine an equitable total chromatic number of splitting graph of \( P_n \), \( C_n \) and \( K_{1,n} \).

Keywords: Equitable total coloring; Equitable total chromatic number; Splitting graph.


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1. Introduction

In this paper, we consider only finite undirected graphs without loops or multiple edges. Let \( G = (V(G), E(G)) \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). In 1994, the concept of total coloring \( \chi''(G) \) was introduced by Behzad [1] and Vizing [11]. A total coloring of a graph \( G \) is an assignment of colors to both the vertices and edges of \( G \), such that no two adjacent or incident vertices and edges of \( G \) are received the same colors. They both conjectured that for any graph \( G \) the following inequality holds:

\[
\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2,
\]

where \( \Delta(G) \) is the maximum degree of \( G \). It is clear that \( \Delta(G) + 1 \) is the possible lower bound. In 1994, Fu [5] first introduced the concept of equitable total coloring and the equitable total chromatic number of a graph. Gong Kun et.al [3] proved some results on the equitable total chromatic number of \( W_n \lor K_n \), \( F_m \lor K_n \) and \( S_m \lor K_n \). In 2012, Ma Gang and Ma Ming [6] proved some results concerning the equitable total chromatic number of \( P_m \lor S_n \), \( P_m \lor F_n \) and \( P_m \lor W_n \). Tong et.al [9] proved that the equitable total chromatic number of \( C_m \square C_n \). Girija et.al [2] proved that equitable total chromatic number of Double star graph and fan graph. Gang et.al [7] proved that on the equitable total coloring of multiple join graph. Zhang Zhong-fu [13] proved that on the equitable total coloring of some join graphs. Veninstine vivik et.al [10] proved an algorithmic approach to equitable total chromatic number of wheel graph, Gear graph, Helm graph and sunlet graph.

2. Preliminaries

**Definition 2.1.** The splitting graph[8] of a graph \( G \) is obtained from adding to each vertex \( v \), a new vertex \( v' \) such that \( v' \) is adjacent to every vertex that is adjacent to \( v \) in \( G \) that is \( N(v) = N(v') \). It is denoted by \( S'(G) \).

**Definition 2.2.** For a simple graph \( G(V,E) \), let \( f \) be a proper \( k \)- total coloring of \( G \).

\[
||T_i| - |T_j|| \leq 1, \text{ for } i, j = 1, 2, \ldots k.
\]

The partition \( \{T_i\} = \{V_i \cup E_i : 1 \leq i \leq k\} \) is called a \( k \)- equitable total coloring and

\[
\chi_{et}(G) = \min \{k/ k\text{-equitable total coloring of } G\}
\]

is called the equitable total chromatic number of \( G \), where for all \( x \in T_i = V_i \cup E_i \), \( f(x) = i, \text{ for } i = 1, 2, \ldots k \).
Conjecture 2.3([5]) For any simple graph \( G(V, E) \),
\[
\chi_{et}(G) \leq \Delta(G) + 2.
\]

Conjecture 2.4([14]) For any simple graph \( G(V, E) \),
\[
\chi_{et}(G) \geq \chi''(G) \geq \Delta(G) + 1.
\]

Conjecture 2.5([12]) For every graph \( G \), \( G \) has an equitable total coloring for each \( k \geq \max\{\chi''(G), \Delta(G) + 2\} \).

Lemma 2.6([4]) For \( n \geq 13 \), the equitable total chromatic number of Hypo-Mycielski Graph, \( \chi_{et}(HM(W_n)) = n + 2 \).

In this paper, we determine an equitable total chromatic number of splitting graph of \( P_n, C_n \) and \( K_{1,n} \).

3. Main Results

Theorem 3.1. For any positive integer \( n \geq 3 \), \( \chi_{et}(S'(P_n)) = 5 \).

Proof. Let \( V(P_n) = \{v_i : 1 \leq i \leq n\} \) and \( E(P_n) = \{e_i : 1 \leq i \leq n - 1\} \), where \( \{e_i : 1 \leq i \leq n - 1\} \) be the edges \( v_iv_{i+1}(1 \leq i \leq n - 1) \). By the definition of splitting graph, introduce the new vertices \( \{v'_i : 1 \leq i \leq n\} \) corresponding to the vertices \( \{v_i : 1 \leq i \leq n\} \) of \( P_n \), which are added to obtain splitting graph of path \( S'(P_n) \). In \( S'(P_n) \), the vertex set and the edge set is given by \( V(S'(P_n)) = \{v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \) and
\[
E(S'(P_n)) = \{e_i : 1 \leq i \leq n - 1\} \cup \{e'_i : 1 \leq i \leq n - 1\} \cup \{e''_i : 1 \leq i \leq n - 1\},
\]
where \( e_i(1 \leq i \leq n - 1) \) is an edge \( v_iv_{i+1}(1 \leq i \leq n - 1) \), \( e'_i(1 \leq i \leq n - 1) \) is an edge \( v_iv'_{i+1}(1 \leq i \leq n - 1) \) and \( e''_i(1 \leq i \leq n - 1) \) is an edge \( v''_iv_{i+1}(1 \leq i \leq n - 1) \). Now we partition the vertex set and edge set of \( S'(P_n) \) as follows. We consider the following two cases.

Case(i): When \( n \) is odd
\[
T_1 = \{v_1, v_3, \ldots, v_n\} \cup \{e''_2, e''_4, \ldots, e''_{n-1}\}
\]
\[
T_2 = \{v_2, v_4, \ldots, v_{n-1}\} \cup \{e''_1, e''_3, \ldots, e''_{n-2}\}
\]
\[
T_3 = \{e_1, e_3, \ldots, e_{n-2}\} \cup \{v'_i : 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \}
\]
\[
T_4 = \{e_2, e_4, \ldots, e_{n-1}\} \cup \{v'_i : \left\lfloor \frac{n}{2} \right\rfloor \leq i \leq n - 1\}
\]
Clearly \( T_1, T_2, T_3, T_4 \) and \( T_5 \) are independent sets of \( S'(P_n) \). Its satisfies the inequality \( ||T_i| - |T_j|| \leq 1 \) for \( i \neq j \). Therefore the graph \( S'(P_n) \) is equitably total colored with 5 colors. This implies that \( \chi_{et}(S'(P_n)) \leq 5 \). Further, since \( \Delta = 4 \), we have \( \chi_{et}(S'(P_n)) \geq 5 \). Hence \( \chi_{et}(S'(P_n)) = 5 \).

**Case(ii):** When \( n \) is even

\[
T_1 = \{v_1, v_3, \ldots, v_{n-1}\} \cup \{e_2, e_4, \ldots, e_{n-2}\}
\]
\[
T_2 = \{v_2, v_4, \ldots, v_n\} \cup \{e_1, e_3, \ldots, e_{n-1}\}
\]
\[
T_3 = \{e_1, e_3, \ldots, e_{n-1}\} \cup \{v_i : 1 \leq i \leq \frac{n}{2}\}
\]
\[
T_4 = \{e_2, e_4, \ldots, e_{n-2}\} \cup \{v_i : \frac{n}{2} + 1 \leq i \leq n\}
\]
\[
T_5 = \{e_i : 1 \leq i \leq n - 1\}
\]

Clearly \( T_1, T_2, T_3, T_4 \) and \( T_5 \) are independent sets of \( S'(P_n) \). Its satisfies the inequality \( ||T_i| - |T_j|| \leq 1 \) for \( i \neq j \). Therefore the graph \( S'(P_n) \) is equitably total colored with 5 colors. This implies that \( \chi_{et}(S'(P_n)) \leq 5 \). Further, since \( \Delta = 4 \), we have \( \chi_{et}(S'(P_n)) \geq 5 \). Hence \( \chi_{et}(S'(P_n)) = 5 \). □

**Theorem 3.2.** For any positive integer \( n \geq 4 \), \( \chi_{et}(S'(C_n)) = 5 \).

**Proof.** Let \( V(C_n) = \{v_i : 1 \leq i \leq n\} \) and \( E(C_n) = \{e_i : 1 \leq i \leq n - 1\} \cup \{e_n\} \), where \( \{e_i : 1 \leq i \leq n - 1\} \) be the edges \( v_iv_{i+1}(1 \leq i \leq n - 1) \) and \( e_n \) is an edge \( v_nv_1 \). By the definition of splitting graph, introduce the new vertices \( \{v'_i : 1 \leq i \leq n\} \) corresponding to the vertices \( \{v_i : 1 \leq i \leq n\} \) of \( C_n \), which are added to obtain splitting graph of cycle \( S'(C_n) \). In \( S'(C_n) \), the vertex set and the edge set is given by

\[
V(S'(C_n)) = \{v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\} \text{ and}
\]
\[
E(S'(C_n)) = \{e_i : 1 \leq i \leq n - 1\} \cup \{e_i : 1 \leq i \leq n - 1\} \cup \{e'_i : 1 \leq i \leq n - 1\} \cup \{e''_i : 1 \leq i \leq n - 1\} + \{e_n\} \cup \{e'_n\} \cup \{e''_n\}, \text{ where} e_i(1 \leq i \leq n - 1) \text{ is an edge} v_iv_{i+1}(1 \leq i \leq n - 1), e_n \text{ is an edge} v_nv_1, e'_i(1 \leq i \leq n - 1) \text{ is an edge} v'_iv_{i+1}(1 \leq i \leq n - 1), e''_i \text{ is an edge} v'_iv_{i+1}(1 \leq i \leq n - 1) \text{ and} e''_n \text{ is an edge} v'_nv_1. \text{ Now we partition the vertex set and edge set of} S'(C_n) \text{ as follows. we consider the following two cases,}

**Case(i):** When \( n \) is odd

\[
T_1 = \{v_1, v_3, \ldots, v_{n-2}\} \cup \{e''_1, e''_3, \ldots, e''_{n-2}\} \cup \{e''_{n-1}\}
\]
Hence be the pendent vertices and splitting graph of star, introduce the new vertices for any positive integer. When Case (ii):

\[
|\Delta| \leq 4 + 1 = 5.
\]

Hence, since \(\Delta = 4\), we have \(\chi_{et}(S'(C_n)) = \chi''(S'(C_n)) \geq \Delta + 1 \geq 4 + 1 = 5\).

**Case (ii):** When \(n\) is even

\[
T_1 = \{v_1, v_3, \ldots, v_{n-1}\} \cup \{e_1', e_3', \ldots, e_{n-1}'\}
\]

\[
T_2 = \{v_2, v_4, \ldots, v_n\} \cup \{e_2', e_4', \ldots, e_n'\}
\]

\[
T_3 = \{e_1, e_3, \ldots, e_{n-1}\} \cup \{v_i' : 1 \leq i \leq \frac{n}{2}\}
\]

\[
T_4 = \{e_2, e_4, \ldots, e_n\} \cup \{v_i' : \frac{n}{2} + 1 \leq i \leq n\}
\]

\[
T_5 = \{e_i' : 1 \leq i \leq n\}
\]

Clearly \(T_1, T_2, T_3, T_4\) and \(T_5\) are independent sets of \(S'(C_n)\). Its satisfies the inequality \(||T_i| - |T_j|| \leq 1\) for \(i \neq j\). Therefore the graph \(S'(C_n)\) is equitably total colored with 5 colors. This implies that \(\chi_{et}(S'(C_n)) \leq 5\). Further, since \(\Delta = 4\), we have \(\chi_{et}(S'(C_n)) \geq \chi''(S'(C_n)) \geq \Delta + 1 \geq 4 + 1 \geq 5\).

Hence \(\chi_{et}(S'(C_n)) = 5\).

**Theorem 3.3.** For any positive integer \(n\), \(\chi_{et}(S'(K_{1,n})) = 2n + 1\), \(n \geq 2\).

**Proof.** Let \(V(K_{1,n}) = \{v\} \cup \{v_i : 1 \leq i \leq n\}\), where \(\{v_i : 1 \leq i \leq n\}\) be the pendent vertices and \(\{v\}\) be the root vertex of \(K_{1,n}\) and \(E(K_{1,n}) = \{e_i : 1 \leq i \leq n\}\), where \(e_i\) is an edge \(vv_i(1 \leq i \leq n)\). Now construct the splitting graph of star, introduce the new vertices \(\{v'\}\) and \(\{v'_i : 1 \leq i \leq n\}\) corresponding to the vertices \(\{v\}\) and \(\{v_i : 1 \leq i \leq n\}\) of \(K_{1,n}\), which are added to obtain splitting graph of star \(S'(K_{1,n})\). In \(S'(K_{1,n})\), the vertex set and the edge sets are given by \(V(S'(K_{1,n})) = \{v_i : 1 \leq i \leq n\} \cup \{v'_i : 1 \leq i \leq n\}\) and \(E(S'(K_{1,n})) = \{e_i : 1 \leq i \leq n\} \cup \{e'_i : 1 \leq i \leq n\}\)
\[ i \leq n \} \cup \{ e_i' : 1 \leq i \leq n \}, \text{where } e_i(1 \leq i \leq n) \text{ is an edge } v v_i(1 \leq i \leq n), \]
\[ e_i'(1 \leq i \leq n) \text{ is an edge } v v_i'(1 \leq i \leq n) \text{ and } e_i''(1 \leq i \leq n) \text{ is an edge } v' v_i(1 \leq i \leq n). \]
Now we partition the vertex set and edge set of \( S'(K_{1,n}) \) as follows.

\[ T_1 = \{ v, v' \} \]
\[ T_2 = \{ e_1', e_1'', v_n \} \]
\[ T_3 = \{ e_2', e_2'', v'_1 \} \]
\[ T_4 = \{ e_3', e_3'', v'_2 \} \]
\[ T_5 = \{ e_4', e_4'', v'_3 \} \]

\[ \cdots \]
\[ T_{n-1} = \{ e_{n-2}', e_{n-2}'', v_{n-3}' \} \]
\[ T_n = \{ e_{n-1}', e_{n-1}''', v_{n-2}' \} \]
\[ T_{n+1} = \{ e_n', e_n'', v'_{n-1} \} \]
\[ T_{n+2} = \{ e_1, v'_n \} \]
\[ T_{n+3} = \{ e_2, v_1 \} \]
\[ T_{n+4} = \{ e_3, v_2 \} \]

\[ \cdots \]
\[ T_{2n-1} = \{ e_{n-2}, v_{n-3} \} \]
\[ T_{2n} = \{ e_{n-1}, v_{n-2} \} \]
\[ T_{2n+1} = \{ e_n, v_{n-1} \} \]

Clearly \( T_1, T_2, T_3, T_4 \ldots T_{2n+1} \) are independent sets of \( S'(K_{1,n}) \). Also \( |T_2| = |T_3| = |T_4| = \ldots = |T_{n+1}| = 3 \) and \( |T_1| = |T_{n+2}| = |T_{n+3}| = \ldots = |T_{2n}| = |T_{2n+1}| = 2 \). Its satisfies the inequality \(||T_i| - |T_j|| \leq 1 \) for \( i \neq j \).

Therefore the graph \( S'(K_{1,n}) \) is equitably total colored with \( 2n+1 \) colors. This implies that \( \chi_{et}(S'(K_{1,n})) \leq 2n + 1 \). Further, since \( \Delta = 2n \), we have \( \chi_{et}(S'(K_{1,n})) \geq \chi'((S'(K_{1,n})) \geq \Delta + 1 \geq 2n + 1 \). Hence \( \chi_{et}(S'(K_{1,n})) = 2n + 1 \). \( \square \)
References


