(p, q)-Lucas polynomials and their applications to bi-univalent functions

Şahsene Altunkaya*  id orcid.org/0000-0002-7950-8450
Sibel Yalçın** id orcid.org/0000-0002-0236-3097
*Bursa Uludağ University, Dept. of Mathematics, Bursa, Turkey.
 sahsenemailkaya@gmail.com
**Bursa Uludağ University, Dept. of Mathematics, Bursa, Turkey.
 syalcin@uludag.edu.tr

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Abstract:

In the present paper, by using the \( L_{p,q}(x) \) functions, our methodology intertwine to yield the Theory of Geometric Functions and that of Special Functions, which are usually considered as very different fields. Thus, we aim at introducing a new class of bi-univalent functions defined through the \((p, q)\)-Lucas polynomials. Furthermore, we derive coefficient inequalities and obtain Fekete-Szegö problem for this new function class.

Keywords: \((p, q)\)-Lucas polynomials; Coefficient bounds; Bi-univalent functions.


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1. Introduction and definitions

Fibonacci polynomials, Lucas polynomials, Lucas-Lehmer polynomials, Chebyshev polynomials, Pell polynomials, Morgan-Voyce polynomials, Orthogonal polynomials and the other special polynomials and their generalizations are of wide spectra in a variety of branches such as Physics, Engineering, Architecture, Nature, Art, Number Theory, Combinatorics and Numerical analysis (see, for example, [8], [10], [11], [12], [14], [15], [16] and [17]).

The well-known \((p,q)\)-Lucas polynomials are defined by the following definition:

Definition 1.1. (see [7]) Let \(p(x)\) and \(q(x)\) be polynomials with real coefficients. The \((p,q)\)-Lucas polynomials \(L_{p,q,n}(x)\) are established by the recurrence relation

\[
L_{p,q,n}(x) = p(x)L_{p,q,n-1}(x) + q(x)L_{p,q,n-2}(x) \quad (n \geq 2),
\]

from which the first few Lucas polynomials can be found as

\[
\begin{align*}
L_{p,q,0}(x) &= 2, \\
L_{p,q,1}(x) &= p(x), \\
L_{p,q,2}(x) &= p^2(x) + 2q(x), \\
L_{p,q,3}(x) &= p^3(x) + 3p(x)q(x), \
\end{align*}
\]

For the special cases of \(p(x)\) and \(q(x)\), we can get the polynomials given in Table 1.

Table 1: Special cases of the \(L_{p,q,n}(x)\) with given initial conditions are given.

<table>
<thead>
<tr>
<th>(p(x))</th>
<th>(q(x))</th>
<th>(L_{p,q,n}(x))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(x)</td>
<td>1</td>
<td>Lucas polynomials (L_n(x))</td>
</tr>
<tr>
<td>(2x)</td>
<td>1</td>
<td>Pell-Lucas polynomials (D_n(x))</td>
</tr>
<tr>
<td>1</td>
<td>(2x)</td>
<td>Jacobsthal-Lucas polynomials (j_n(x))</td>
</tr>
<tr>
<td>3x</td>
<td>-2</td>
<td>Fermat-Lucas polynomials (f_n(x))</td>
</tr>
<tr>
<td>2x</td>
<td>-1</td>
<td>Chebyshev polynomials first kind (T_n(x))</td>
</tr>
</tbody>
</table>

Theorem 1.1. (see [7]) Let \(G_{\{L_{p,q,n}(x)\}}(z)\) be the generating function of the \((p,q)\)-Lucas polynomial sequence \(L_{p,q,n}(x)\). Then

\[
G_{\{L_{p,q,n}(x)\}}(z) = \sum_{n=0}^{\infty} L_{p,q,n}(x)z^n = \frac{2 - p(x)z}{1 - p(x)z - q(x)z^2}.
\]
Let $A$ be the class of functions $f$ of the form

\[(1.2) \quad f(z) = z + a_2 z^2 + a_3 z^3 + \cdots,\]

which are analytic in the open unit disk $\Delta = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$ and normalized under the condition $f(0) = f'(0) - 1 = 0$. Further, by $S$ we represent the class of all functions in $A$ which are univalent in $\Delta$.

With a view to recalling the principle of subordination between analytic functions, let the functions $f$ and $g$ be analytic in $\Delta$. Given functions $f, g \in A$, $f$ is subordinate to $g$ if there exists a Schwarz function $w \in \Lambda$, where

\[\Lambda = \{w : w(0) = 0, \ |w(z)| < 1, \ z \in \Delta\},\]

such that

\[f(z) = g(w(z)) \quad (z \in \Delta).\]

We show this subordination by

\[f \prec g \quad \text{or} \quad f(z) \prec g(z) \quad (z \in \Delta).\]

In particular, if the function $g$ is univalent in $\Delta$, the above subordination is equivalent to

\[f(0) = g(0), \quad f(\Delta) \subset g(\Delta).\]

According to the Koebe-One Quarter Theorem [4], it ensures that the image of $\Delta$ under every univalent function $f \in A$ contains a disc of radius $1/4$. Thus every univalent function $f \in A$ has an inverse $f^{-1}$ satisfying $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right)$, where

\[f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right) w^4 + \cdots.\]

\[(1.3)\]

A function $f \in A$ is said to be bi-univalent in $\Delta$ if both $f$ and $f^{-1}$ are univalent in $\Delta$. Let $\Sigma$ indicate the class of bi-univalent functions in $\Delta$ given
by (1.2). For a brief history and interesting examples in the class $\Sigma$, see [13] (see also [1], [2], [3], [6] and [9]).

In the present paper, by using the $L_{p,q,n}(x)$ functions, our methodology intertwine to yield the Theory of Geometric Functions and that of Special Functions, which are usually considered as very different fields. Thus, we aim at introducing a new class of bi-univalent functions defined through the $(p,q)$-Lucas polynomials. Furthermore, we derive coefficient inequalities and obtain Fekete-Szeg problem for this new function class.

**Definition 1.2.** A function $f \in \Sigma$ is said to be in the class

$$W_\Sigma(\tau, \mu, \eta; x) \quad (\tau \in \mathbb{C}\backslash\{0\}, \mu \geq 0, \eta \geq 0; \ z, w \in \Delta)$$

if the following subordinations are satisfied:

$$\left[ 1 + \frac{1}{\tau} \left( (1 - \mu + 2\eta) \frac{f(z)}{z} + (\mu - 2\eta) f'(z) + \eta z f''(z) - 1 \right) \right] \prec G_{\{L_{p,q,n}(x)\}}(z) - 1$$

and

$$\left[ 1 + \frac{1}{\tau} \left( (1 - \mu + 2\eta) \frac{g(w)}{w} + (\mu - 2\eta) g'(w) + \eta w g''(w) - 1 \right) \right] \prec G_{\{L_{p,q,n}(x)\}}(w) - 1$$

where the function $g$ is given by (1.3).

It is interesting to note that the special values of $\tau, \mu$ and $\eta$ lead the class $W_\Sigma(\tau, \mu, \eta; x)$ to various subclasses, we illustrate the following subclasses:

1. For $\mu = 1 + 2\eta$, we get the class $W_\Sigma(\tau, 1 + 2\eta, \eta; x) = W_\Sigma(\tau, \eta; x)$. A function $f \in \Sigma$ is said to be in the class

$$W_\Sigma(\tau, \eta; x) \quad (\tau \in \mathbb{C}\backslash\{0\}, \mu \geq 0, \ z, w \in \Delta)$$

if the following subordinations are satisfied:

$$\left[ 1 + \frac{1}{\tau} \left( f'(z) + \eta z f''(z) - 1 \right) \right] \prec G_{\{L_{p,q,n}(x)\}}(z) - 1$$

and

$$\left[ 1 + \frac{1}{\tau} \left( g'(w) + \eta w g''(w) - 1 \right) \right] \prec G_{\{L_{p,q,n}(x)\}}(w) - 1$$

where the function $g$ is given by (1.3).
2. For $\eta = 0$, we obtain the class $W_\Sigma (\tau, \mu, 0; x) = W_\Sigma (\tau, \mu; x)$. A function $f \in \Sigma$ is said to be in the class

$$W_\Sigma (\tau, \mu; x) \quad (\tau \in C \setminus \{0\}, \mu \geq 0; \ z, w \in \Delta)$$

if the following subordinations are satisfied:

$$\left[ 1 + \frac{1}{\tau} \left( (1 - \mu) \frac{f(z)}{z} + \mu zf'(z) - 1 \right) \right] \prec G_{\{L_{p,q,n}(x)\}}(z) - 1$$

and

$$\left[ 1 + \frac{1}{\tau} \left( (1 - \mu) \frac{g(w)}{w} + \mu g'(w) - 1 \right) \right] \prec G_{\{L_{p,q,n}(x)\}}(w) - 1$$

where the function $g$ is given by (1.3).

3. For $\eta = 0$ and $\mu = 1$, we get the class $W_\Sigma (\tau, 1, 0; x) = W_\Sigma (\tau; x)$. A function $f \in \Sigma$ is said to be in the class

$$W_\Sigma (\tau, \mu; x) \quad (\tau \in C \setminus \{0\}; \ z, w \in \Delta)$$

if the following subordinations are satisfied:

$$\left[ 1 + \frac{1}{\tau} (f'(z) - 1) \right] \prec G_{\{L_{p,q,n}(x)\}}(z) - 1$$

and

$$\left[ 1 + \frac{1}{\tau} (g'(w) - 1) \right] \prec G_{\{L_{p,q,n}(x)\}}(w) - 1$$

where the function $g$ is given by (1.3).

2. Coefficient bounds

In this section, we shall make use of the $(p, q)$-Lucas polynomials to get the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $W_\Sigma (\tau, \mu, \eta; x)$ proposed by Definition 1.2.
**Theorem 2.1.** Let $f$ given by (1.2) be in the class $W_{\Sigma} (\tau, \mu, \eta; x)$. Then

$$|a_2| \leq \frac{|\tau| p(x) |\sqrt{p(x)}|}{\sqrt{[1 + 2(2\mu + 2\eta)\tau - (1 + \mu)^2] p^2(x) - 2(1 + \mu)^2 q(x)}}$$

and

$$|a_3| \leq \frac{|\tau|^2 p^2(x)}{(1 + \mu)^2} + \frac{|\tau| p(x)}{1 + 2\mu + 2\eta}.$$ 

**Proof.** Let $f \in W_{\Sigma} (\tau, \mu, \eta; x)$. From Definition 1.2, for some analytic functions $\Phi, \Psi$ such that $\Phi(0) = \Psi(0) = 0$ and $|\Phi(z)| < 1, |\Psi(w)| < 1$ for all $z, w \in \Delta$, we can write

$$1 + \frac{1}{\tau} \left( (1 - \mu + 2\eta) \frac{f(z)}{z} + (\mu - 2\eta) f'(z) + \eta z f''(z) - 1 \right) = G_{\{L_{p,q,n}(x)\}}(\Phi(z)) - 1,$$

and

$$1 + \frac{1}{\tau} \left( (1 - \mu + 2\eta) \frac{g(w)}{w} + (\mu - 2\eta) g'(w) + \eta w g''(w) - 1 \right) = G_{\{L_{p,q,n}(x)\}}(\Psi(w)) - 1,$$

or equivalently

$$1 + \frac{1}{\tau} \left( (1 - \mu + 2\eta) \frac{f(z)}{z} + (\mu - 2\eta) f'(z) + \eta z f''(z) - 1 \right) = -1 + L_{p,q,0}(x) + L_{p,q,1}(x) \Phi(z) + L_{p,q,2}(x) \Phi^2(z) + \cdots.$$ 

(2.1)

$$1 + \frac{1}{\tau} \left( (1 - \mu + 2\eta) \frac{g(w)}{w} + (\mu - 2\eta) g'(w) + \eta w g''(w) - 1 \right) = -1 + L_{p,q,0}(x) + L_{p,q,1}(x) \Psi(w) + L_{p,q,2}(x) \Psi^2(w) + \cdots.$$ 

(2.2)

From the equalities (2.1) and (2.2), we obtain that

$$1 + \frac{1}{\tau} \left( (1 - \mu + 2\eta) \frac{f(z)}{z} + (\mu - 2\eta) f'(z) + \eta z f''(z) - 1 \right) = 1 + L_{p,q,1}(x) t_1 z + \left[ L_{p,q,1}(x) t_2 + L_{p,q,2}(x) t_2^2 \right] z^2 + \cdots,$$

and
\[ 1 + \frac{1}{\tau} \left( (1 - \mu + 2\eta) \frac{g(w)}{w} + (\mu - 2\eta)g'(w) + \eta wg''(w) - 1 \right) \]
\[ = 1 + L_{p,q,1}(x)s_1 w + \left[ L_{p,q,1}(x)s_2 + L_{p,q,2}(x)s_1^2 \right] w^2 + \cdots. \]  

(2.4)

It is fairly well-known that if

\[ |\Phi(z)| = |t_1 z + t_2 z^2 + t_3 z^3 + \cdots| < 1 \quad (z \in \Delta) \]

and

\[ |\Psi(w)| = |s_1 w + s_2 w^2 + s_3 w^3 + \cdots| < 1 \quad (w \in \Delta), \]

then

(2.5) \[ |t_k| \leq 1 \quad \text{and} \quad |s_k| \leq 1 \quad (k \in \mathbb{N}). \]

Thus, upon comparing the corresponding coefficients in (2.3) and (2.4), we have

(2.6) \[ \frac{1}{\tau} (1 + \mu)a_2 = L_{p,q,1}(x)t_1, \]

(2.7) \[ \frac{1}{\tau} (1 + 2\mu + 2\eta)a_3 = L_{p,q,1}(x)t_2 + L_{p,q,2}(x)t_1^2, \]

(2.8) \[ -\frac{1}{\tau} (1 + \mu)a_2 = L_{p,q,1}(x)s_1 \]

and

(2.9) \[ \frac{1}{\tau} (1 + 2\mu + 2\eta) \left( 2a_2^2 - a_3 \right) = L_{p,q,1}(x)s_2 + L_{p,q,2}(x)s_1^2. \]

From the equations (2.6) and (2.8), we can easily see that

(2.10) \[ t_1 = -s_1, \]

(2.11) \[ \frac{2}{\tau^2} (1 + \mu)^2 a_2^2 = L_{p,q,1}^2(x) \left( t_1^2 + s_1^2 \right). \]

If we add (2.7) to (2.9), we get
\[(2.12) \quad \frac{2}{\tau}(1 + 2\mu + 2\eta) a_2^2 = L_{p,q,1}(x)(t_2 + s_2) + L_{p,q,2}(x)(t_1^2 + s_1^2). \]

Clearly, by using (2.11) in the equality (2.12), we have

\[
\frac{2 \left[ (1 + 2\mu + 2\eta) \tau L_{p,q,1}^2(x) - (1 + \mu)^2 L_{p,q,2}(x) \right]}{\tau^2 L_{p,q,1}^2(x)} a_2^2 = L_{p,q,1}(x)(t_2 + s_2). \]

\[(2.13)\]

which gives

\[
|a_2| \leq \frac{|\tau| |p(x)| \sqrt{|p(x)|}}{\sqrt{|(1 + 2\mu + 2\eta) \tau - (1 + \mu)^2 p^2(x) - 2 (1 + \mu)^2 q(x)|}}.
\]

Moreover, if we subtract (2.9) from (2.7), we obtain

\[
\frac{2}{\tau}(1 + 2\mu + 2\eta)(a_3 - a_2^2) = L_{p,q,1}(x)(t_2 - s_2) + L_{p,q,2}(x)(t_1^2 - s_1^2).
\]

\[(2.14)\]

Then, in view of (2.10) and (2.11), (2.14) becomes

\[
a_3 = \frac{\tau^2 L_{p,q,1}^2(x)(t_2^2 + s_1^2)}{2 (1 + \mu)^2} + \frac{\tau L_{p,q,1}(x)(t_2 - s_2)}{2(1 + 2\mu + 2\eta)}. \]

It is seen from (1.1) and (2.5) that

\[
|a_3| \leq \frac{|\tau|^2 p^2(x)}{(1 + \mu)^2} + \frac{|\tau| |p(x)|}{1 + 2\mu + 2\eta}.
\]
Corollary 2.1. Let $f$ given by (1.2) be in the class $W_{\Sigma}(\tau, \eta; x)$. Then

$$|a_2| \leq \frac{|\tau| |p(x)| \sqrt{|p(x)|}}{\sqrt{\left| \frac{3(1+2\eta)\tau - 4(1+\eta)^2}{3(1+2\eta)^2} p^2(x) - 8(1+\eta)^2 q(x) \right|}}$$

and

$$|a_3| \leq \frac{|\tau|^2 p^2(x)}{4(1+\eta)^2} + \frac{|\tau| |p(x)|}{3(1+2\eta)}.$$

Corollary 2.2. Let $f$ given by (1.2) be in the class $W_{\Sigma}(\tau, \mu; x)$. Then

$$|a_2| \leq \frac{|\tau| |p(x)| \sqrt{|p(x)|}}{\sqrt{\left| \frac{(1+2\mu)\tau - (1+\mu)^2}{1+2\mu} p^2(x) - 2(1+\mu)^2 q(x) \right|}}$$

and

$$|a_3| \leq \frac{|\tau|^2 p^2(x)}{(1+\mu)^2} + \frac{|\tau| |p(x)|}{1+2\mu}.$$

Corollary 2.3. Let $f$ given by (1.2) be in the class $W_{\Sigma}(\tau; x)$. Then

$$|a_2| \leq \frac{|\tau| |p(x)| \sqrt{|p(x)|}}{\sqrt{|(3\tau - 4)p^2(x) - 8q(x)|}}$$

and

$$|a_3| \leq \frac{|\tau|^2 p^2(x)}{4} + \frac{|\tau| |p(x)|}{3}.$$

3. Fekete-Szeg problem

The classical Fekete-Szeg inequality, presented by means of Loewner’s method, for the coefficients of $f \in S$ is

$$|a_3 - \xi a_2^2| \leq 1 + 2 \exp(-2\xi/(1 - \xi)) \text{ for } \xi \in [0, 1].$$

As $\xi \to 1^-$, we have the elementary inequality $|a_3 - a_2^2| \leq 1$. Moreover, the coefficient functional

$$\Gamma_\xi(f) = a_3 - \xi a_2^2$$
on the normalized analytic functions $f$ in the unit disk $\Delta$ plays an important role in function theory. The problem of maximizing the absolute value of the functional $\Gamma_\xi(f)$ is called the Fekete-Szegő problem, see [5].

In this section, we aim to provide Fekete-Szegő inequalities for functions in the class $T_n^\tau (\tau; x)$. These inequalities are given in the following theorem.

**Theorem 3.1.** Let $f$ given by (1.2) be in the class $W_\Sigma (\tau, \mu, \eta; x)$ and $\xi \in \mathbb{R}$. Then

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{|p(x)|}{(1 + 2\mu + 2\eta) |\tau|^2}, \\ |1 - \xi| \leq \frac{1}{\tau^2} - \frac{(1 + \mu)^2}{(1 + 2\mu + 2\eta) \tau^3} \left(1 + \frac{2q(x)}{p^2(x)}\right) \\ |1 - \xi| \geq \frac{1}{\tau^2} - \frac{(1 + \mu)^2}{(1 + 2\mu + 2\eta) \tau^3} \left(1 + \frac{2q(x)}{p^2(x)}\right) \end{cases}$$

**Proof.** From (2.13) and (2.14)

$$a_3 - \xi a_2^2 = \frac{\tau^3 L_{p,q,1}(x) (1 - \xi) (t_2 + s_2)}{2 (1 + 2\mu + 2\eta) \tau^2 L_{p,q,1}(x) - (1 + \mu)^2 L_{p,q,2}(x)}$$

$$+ \frac{\tau L_{p,q,1}(x) (t_2 - s_2)}{2(1 + 2\mu + 2\eta)}$$

$$= L_{p,q,1}(x) \left[ \left(K(\xi, x) + \frac{1}{2(1 + 2\mu + 2\eta) \tau} \right) t_2 \right. \left. + \left(K(\xi, x) - \frac{1}{2(1 + 2\mu + 2\eta) \tau} \right) s_2 \right]$$

where

$$K(\xi, x) = \frac{\tau^2 L_{p,q,1}^2(x) (1 - \xi)}{2 \left(1 + 2\mu + 2\eta\right) \tau^2 L_{p,q,1}(x) - (1 + \mu)^2 L_{p,q,2}(x)}.$$

Along the way, in view of (1.1), we conclude that
$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{|p(x)|}{(1 + 2\mu + 2\eta)\tau}, & 0 \leq |K(\xi, x)| \leq \frac{1}{2(1 + 2\mu + 2\eta)\tau} \\ 2|p(x)||K(\xi, x)|, & |K(\xi, x)| \geq \frac{1}{2(1 + 2\mu + 2\eta)\tau} \end{cases}$

\[\Box\]

**Corollary 3.1.** Let $f$ given by (1.2) be in the class $W_{\Sigma}(\tau, \eta; x)$ and $\xi \in \mathbb{R}$. Then

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{|p(x)|}{3(1 + 2\eta)\tau} \\ |1 - \xi| \leq \frac{1}{\tau^2} - \frac{4(1 + \eta)^2}{3(1 + 2\eta)\tau^2} \left(1 + \frac{2q(x)}{p^2(x)}\right) \\ \left|\frac{|\tau^2 |p^3(x)| |1 - \xi|}{|3(1 + 2\eta)\tau - 4(1 + \eta)^2 |p^2(x)| - 8(1 + \eta)^2 q(x)|}\right| \\ |1 - \xi| \geq \frac{1}{\tau^2} - \frac{4(1 + \eta)^2}{3(1 + 2\eta)\tau^2} \left(1 + \frac{2q(x)}{p^2(x)}\right) \end{cases}.$$ 

**Corollary 3.2.** Let $f$ given by (1.2) be in the class $W_{\Sigma}(\tau, \mu; x)$ and $\xi \in \mathbb{R}$. Then

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{|p(x)|}{(1 + 2\mu)\tau} \\ |1 - \xi| \leq \frac{1}{\tau^2} - \frac{(1 + \mu)^2}{(1 + 2\mu)\tau^2} \left(1 + \frac{2q(x)}{p^2(x)}\right) \\ \left|\frac{|\tau^2 |p^3(x)| |1 - \xi|}{|(1 + 2\mu)\tau - (1 + \mu)^2 |p^2(x)| - 2(1 + \mu)^2 q(x)|}\right| \\ |1 - \xi| \geq \frac{1}{\tau^2} - \frac{(1 + \mu)^2}{(1 + 2\mu)\tau^2} \left(1 + \frac{2q(x)}{p^2(x)}\right) \end{cases}.$$ 

**Corollary 3.3.** Let $f$ given by (1.2) be in the class $W_{\Sigma}(\tau; x)$ and $\xi \in \mathbb{R}$. Then

$$|a_3 - \xi a_2^2| \leq \begin{cases} \frac{|p(x)|}{(1 + 2\mu + 2\eta)\tau}, & 0 \leq |K(\xi, x)| \leq \frac{1}{2(1 + 2\mu + 2\eta)\tau} \\ 2|p(x)||K(\xi, x)|, & |K(\xi, x)| \geq \frac{1}{2(1 + 2\mu + 2\eta)\tau} \end{cases}.$$
\[
|a_3 - \xi a_2^2| \leq \left\{ \begin{array}{l}
\frac{|p(x)|}{3|\tau|}, \\
|1 - \xi| \leq \frac{1}{\tau^2} - \frac{4}{3\tau^3} \left( 1 + \frac{2q(x)}{p^2(x)} \right)
\end{array} \right.
\]

If we choose $\xi = 1$, we get the next corollaries.

**Corollary 3.4.** If $f \in W_{22}(\tau, \mu, \eta; x)$, then

\[
|a_3 - a_2^2| \leq \frac{|p(x)|}{(1 + 2\mu + 2\eta)|\tau|}.
\]

**Corollary 3.5.** If $f \in W_{22}(\tau, \eta; x)$, then

\[
|a_3 - a_2^2| \leq \frac{|p(x)|}{3(1 + 2\eta)|\tau|}.
\]

**Corollary 3.6.** If $f \in W_{22}(\tau, \mu; x)$, then

\[
|a_3 - a_2^2| \leq \frac{|p(x)|}{(1 + 2\mu)|\tau|}.
\]

**Corollary 3.7.** If $f \in W_{22}(\tau; x)$, then

\[
|a_3 - a_2^2| \leq \frac{|p(x)|}{3|\tau|}.
\]

**References**


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