Nondifferentiable higher-order duality theorems for new type of dual model under generalized functions

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Abstract:

The motivation behind this article is to study a class of nondifferentiable multiobjective fractional programming problem in which each component of objective functions contains a term including the support function of a compact convex set. For a differentiable function, we consider a class of higher order pseudo quasi/strictly pseudo quasi/weak strictly pseudo quasi- \((V, p, d)\)-type-I convex functions. Under these the higher-order pseudo quasi/strictly pseudo quasi/weak strictly pseudo quasi- \((V, p, d)\)-type-I convexity assumptions, we prove the higher-order weak, higher-order strong and higher-order converse duality theorems related to efficient solution. establish Jensen-type and Hermite-Hadamard-type inequalities.

Keywords: Fractional programming; Multiobjective; Support function; Efficient solutions.

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1. Introduction

The fractional optimization problem with multiple objective functions have been the subject of intense investigations in the past few years, which have produced a number of optimality and duality results for these problems. Higher-order duality in non-linear programming has been studied in last few years by many researchers. In various numerical algorithms, higher-order duality is considered over first-order as it gives more closer bounds.

Higher-order duality in nonlinear programs have been studied by some researchers. Mangasarian [8] formulated a class of higher-order dual problems for the nonlinear programming problem "\( \min \{ f(x) : g(x) \geq 0 \} \)" by introducing twice differentiable function \( h : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \) and \( k : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m \). The concept of higher-order convexity presented by Zhang [12] and derived duality results in multiobjective programming problem. Later on, Yang et al. [11] considered a unified higher-order dual model for nondifferentiable multiobjective programs and proved duality results under generalized assumptions. Suneja et al. [10] introduced a higher order \((F,\alpha,\sigma)\)-type I functions and formulated higher order dual programs for a nondifferentiable multiobjective fractional programming problem.

Motivated by several concepts of generalized convexity, Gulati and Agarwal [4] gave the concept of second-order \(-V\)-type I functions for multiobjective programming problem which were recently extended to nondifferentiable case by Jayswal et al. [6]. Recently, Jayswal et al. [7] considered higher-Order duality for multiobjective programming problems and discussed duality theorems under \((F,\alpha,\rho,d) - V\) -- type-I functions. Several researchers [[1], [2], [3], [9], [13]-[17]] have done their work in the related areas.

In this paper, we have generalized the definitions of higher-order pseudo quasi/ strictly pseudo quasi/weak strictly pseudo quasi- \((V,\rho,d)\)-type-I functions for a nondifferentiable multiobjective higher-order fractional programming problem. We have formulated higher-order unified dual and established duality results under higher order pseudo quasi/ strictly pseudo quasi/weak strictly pseudo quasi- \((V,\rho,d)\)-type-I assumptions.
2. Preliminaries and Definitions

Throughout this paper, we use the index sets $K = \{1, 2, ..., k\}$ and $M = \{1, 2, ..., m\}$.

**Definition 2.1.** Let $Q \subseteq \mathbb{R}^n$ be a compact convex set. The support function of $Q$ is defined by

$$s(y|Q) = \max\{y^T z : z \in Q\}.$$ 

Consider the following nondifferentiable multiobjective fractional programming problem:

$$(MFP) \quad \text{Minimize } \Psi(x) = (\phi_1(x) + s(x|C_1), \phi_2(x) + s(x|C_2), \ldots, \phi_k(x) + s(x|C_k))^T$$

subject to $x \in Y^0 = \{x \in Y : \pi_j(x) + s(x|D_j) \leq 0, j \in M\}$,

where $Y \subseteq \mathbb{R}^n$ is an open set. The functions $\phi, \psi : Y \rightarrow \mathbb{R}^k$, $\pi : Y \rightarrow \mathbb{R}^m$ are differentiable on $Y$ and $C_i, E_i, D_j$ are compact convex sets in $\mathbb{R}^n$ for $i \in K$ and $j \in M$. Let $\phi_i(x) + s(x|C_i) \geq 0$ and $\psi_i(x) - s(x|E_i) > 0$, $i \in K$. Let $H = (H_1, H_2, \ldots, H_k) : X \times \mathbb{R}^n \rightarrow \mathbb{R}^k$ and $K = (K_1, K_2, \ldots, K_m) : X \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ be differentiable functions, $d : X \times X \rightarrow R$, $z = (z_1, z_2, \ldots, z_k)$, $v = (v_1, v_2, \ldots, v_k)$ and $w = (w_1, w_2, \ldots, w_m)$, where $z_i \in C_i$, $v_i \in E_i$ and $w_j \in D_j$, for $i \in K$ and $j \in M$. Let $\rho = (\rho^1, \rho^2)$ such that $\rho^1 = (\rho_1, \rho_2, \ldots, \rho_k) \in R^k$, $\rho^2 = (\rho_{k+1}, \rho_{k+2}, \ldots, \rho_{k+m}) \in R^m$.

**Definition 2.2.** A point $u \in Y^0$ is efficient solution of $(MFP)$ if $\exists$ no $x \in Y^0$ such that $\Psi(x) \leq \Psi(u)$.

**Definition 2.3.** $\forall i \in K, j \in M$, $\left(\phi_i(\cdot) + (\cdot)^T z_i, \psi_i(\cdot) - (\cdot)^T v_i, \pi_j(\cdot) + (\cdot)^T w_j\right)$ is higher order pseudo quasi $(V, \rho, d)$-type I at $u$ of $(MFP)$ if, $\forall x \in Y^0$ and $p \in \mathbb{R}^n$, such that
\[
\frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} < \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u, p) - p^T \nabla_p H_i(u, p)
\]

\[
\Rightarrow \eta^T(x, u) \left\{ \nabla \left( \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} \right) + \nabla_p H_i(u, p) \right\} + \rho_i^1 d^2(x, u) < 0
\]

and

\[
-\pi_j(u) - u^T w_j \leq K_j(u, p) - p^T \nabla_p K_j(u, p)
\]

\[
\Rightarrow \eta^T(x, u) \{ \nabla \pi_j(u) + \nabla_p K_j(u, p) \} + \rho_j^2 d^2(x, u) \leq 0.
\]

**Definition 2.4.** \( \forall i \in K, j \in M, \left( \frac{\phi_i(\cdot) + (\cdot)^T z_i}{\psi_i(\cdot) - (\cdot)^T v_i}, \pi_j(\cdot) + (\cdot)^T w_j \right) \) is higher order strictly pseudo quasi \((V, \rho, d)\)-type I at \( u \) of (MFP) if, \( \forall x \in Y^0 \) and \( p \in \mathbb{R}^n \), such that

\[
\frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} \leq \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u, p) - p^T \nabla_p H_i(u, p)
\]

\[
\Rightarrow \eta^T(x, u) \left\{ \nabla \left( \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} \right) + \nabla_p H_i(u, p) \right\} + \rho_i^1 d^2(x, u) < 0
\]

and

\[
-\pi_j(u) - u^T w_j \leq K_j(u, p) - p^T \nabla_p K_j(u, p)
\]

\[
\Rightarrow \eta^T(x, u) \{ \nabla \pi_j(u) + \nabla_p K_j(u, p) \} + \rho_j^2 d^2(x, u) \leq 0.
\]

**Definition 2.5.** \( \forall i \in K, j \in M, \left( \frac{\phi_i(\cdot) + (\cdot)^T z_i}{\psi_i(\cdot) - (\cdot)^T v_i}, \pi_j(\cdot) + (\cdot)^T w_j \right) \) is higher order weak strictly pseudo quasi \((V, \rho, d)\)-type I at \( u \) of (MFP) if, \( \forall x \in Y^0 \) and \( p \in \mathbb{R}^n \), such that

\[
\frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} \leq \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u, p) - p^T \nabla_p H_i(u, p)
\]
Nondifferentiable higher-order duality theorems for new type of ...

\[ \Rightarrow \eta^T(x, u) \left\{ \nabla \left( \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} \right) + \nabla_p H_i(u, p) \right\} + \rho_1^2 d^2(x, u) < 0 \]

and

\[
-\pi_j(u) - u^T w_j \leq K_j(u, p) - p^T \nabla_p K_j(u, p) \\
\Rightarrow \eta^T(x, u) \{ \nabla \pi_j(u) + \nabla_p K_j(u, p) \} + \rho_2^2 d^2(x, u) \leq 0.
\]

**Definition 2.6.** \( \forall i \in K, j \in M, \left( \frac{\phi_i(.) + (.)^T z_i}{\psi_i(.) - (.)^T v_i}, \pi_j(.) + (.)^T w_j \right) \) is higher order quasi strictly pseudo \((V, \rho, d)-type I\) at \( u \) of \((MFP)\) if, \( \forall x \in Y^0 \) and \( p \in R^n \), such that

\[
\frac{\phi_i(x) + x^T z_i}{\psi_i(x) - x^T v_i} \leq \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} + H_i(u, p) - p^T \nabla_p H_i(u, p) \\
\Rightarrow \eta^T(x, u) \left\{ \nabla \left( \frac{\phi_i(u) + u^T z_i}{\psi_i(u) - u^T v_i} \right) + \nabla_p H_i(u, p) \right\} + \rho_1^2 d^2(x, u) \leq 0
\]

and

\[
-\pi_j(u) - u^T w_j \leq K_j(u, p) - p^T \nabla_p K_j(u, p) \\
\Rightarrow \eta^T(x, u) \{ \nabla \pi_j(u) + \nabla_p K_j(u, p) \} + \rho_2^2 d^2(x, u) < 0.
\]

**Theorem 2.1** \((K-K-T-type necessary condition)\)[5]. Let \( u \) be efficient solution of \((MFP)\) at which the Kuhn-Tucker constraint qualification is satisfied on \( X \). Then, \( \exists 0 < \lambda \in R^k, 0 \leq \bar{y}_j \in R^m, \bar{z}_i \in R^n, \bar{v}_i, \bar{w}_j \in R^n, i \in K, j \in M \) such that

\[
(2.1) \quad \sum_{i=1}^k \lambda_i \nabla \left( \frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right) + \sum_{j=1}^m \bar{y}_j \nabla (\pi_j(u) + u^T \bar{w}_j) = 0,
\]

\[
(2.2) \quad \sum_{j=1}^m \bar{y}_j (\pi_j(u) + u^T \bar{w}_j) = 0,
\]
(2.3) \[ u^T z_i = S(u|C_i), \quad u^T \bar{v}_i = S(u|E_i), \quad u^T \bar{w}_j = S(u|D_j), \]

(2.4) \[ \bar{z}_i \in C_i, \quad \bar{v}_i \in D_i, \quad \bar{w}_j \in E_j, \quad i \in K, \quad j \in M. \]

In the following section, we consider the following mixed higher-order dual for (MFP) and derive duality theorems. The notation \[ \frac{\phi(.) + (.)^T z_i}{\psi(.) - (.)^T v_i} e \] denotes the vector whose components are
\[ \frac{\phi_1(.) + (.)^T z_1}{\psi_1(.) - (.)^T v_1} + \sum_{j \in J_0} \mu_j \left( \pi_j + (.)^T w_j \right), \]
\[ \frac{\phi_2(.) + (.)^T z_2}{\psi_2(.) - (.)^T v_2} + \sum_{j \in J_0} \mu_j \left( \pi_j + (.)^T w_j \right), \ldots, \frac{\phi_k(.) + (.)^T z_k}{\psi_k(.) - (.)^T v_k} + \sum_{j \in J_0} \mu_j \left( \pi_j + (.)^T w_j \right) \]
and \{\pi + (.)^T w\}_{J_0}^\mu denotes r-dimensional vector whose components are
\[ \sum_{j \in J_1} \{\pi_j + (.)^T w_j\}, \quad \sum_{j \in J_2} \{\pi_j + (.)^T w_j\}, \ldots, \sum_{j \in J_r} \{\pi_j + (.)^T w_j\}. \]

3. Unified higher-order duality model:

In this section, we formulate the following unified higher-order dual for (MFP) and establish duality theorems:

(HMDP) : Maximize \[ \left( \phi_1(y) + y^T z_1 \right) + H_1(y, p) - p^T \nabla_p H_1(y, p) + \sum_{j \in J_0} \mu_j \left\{ \pi_j(y) + y^T \bar{w}_j - K_j(y, p) + p^T \nabla_p K_j(y, p) \right\} \]
subject to \[ y \in Y, \]

(3.1) \[ \sum_{i=1}^k \lambda_i \left\{ \nabla \left( \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \]
Nondifferentiable higher-order duality theorems for new type of ... 21

\[ + \sum_{j=1}^{m} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} = 0, \]

(3.2) \[ \sum_{j \in J_\beta} \mu_j \{ \pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p) \} \geq 0, \quad \beta = 1, ..., r, \]

(3.3) \[ \lambda_i \geq 0, \quad \sum_{i=1}^k \lambda_i = 1, \]

(3.4) \[ \mu_j \geq 0, \quad z_i \in C_i, \quad v_i \in E_i, \quad w_j \in D_j \quad \text{for } i \in K, \quad j \in M, \]

where \( J_\delta \subseteq N, \delta = 0, 1, ..., r \) with \( \bigcup_{0=0}^{r} J_\delta = N \) and \( J_{\delta_1} \cap J_{\delta_2} = \emptyset \) if \( \delta_1 \neq \delta_2 \).

It may be noted that if \( J_0 = N \), \( J_1 = N \) and \( J_\beta = \emptyset \) (\( 1 \leq \beta \leq r \)), we obtain Wolfe type dual. If \( J_0 = \emptyset, \quad J_1 = N \) and \( J_\beta = \emptyset \) (\( 2 \leq \beta \leq r \)), then (HMDP) reduces to Mond-Weir Type dual.

Let \( W^0 \) be feasible solution of (HMDP).

**Theorem 3.1 (Weak Duality Theorem).** Let \( x \in Y^0 \) and \( (y, \lambda, v, \mu, z, w, p) \in W^0 \). Let

(i) \[ \left( \frac{\phi_i(.) + (.)^T z_i}{\psi_i(.) - (.)^T v_i} + \mu_{J_0} \left( \pi_j J_0 + (.)^T w_j J_0 \right) \right)e, \{ \pi_j(.) + (.)^T w_j \} \]

(ii) \[ k \sum_{i=1}^k \lambda_i \rho_i^1 + \sum_{j=1}^r \mu_j \rho_j^2 \geq 0. \]

Then, the following cannot hold

\[ \frac{\phi_i(x) + s(x|C_i)}{\psi_i(x) - s(x|E_i)} \leq \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} + H_i(y, p) - p^T \nabla_p H_i(y, p) \]

(3.5) \[ + \sum_{j \in J_0} \mu_j \{ \pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p) \}, \forall i \in K \]
and
\[
\frac{\phi_j(x) + s(x|C_j)}{\psi_j(x) - s(x|E_j)} < \frac{\phi_j(y) + y^T z_j}{\psi_j(x) - y^T v_j} + H_j(y, p) - p^T \nabla_p H_j(y, p) \\
+ \sum_{j \in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p)\}, \text{ for some } j \in K.
\] (3.6)

**Proof:** If possible, then suppose inequalities (3.5) and (3.6) hold. As
\[x^T z_i \leq s(x|C_i), \quad x^T v_i \leq s(x|E_i), \quad \forall i \in K\]
and \[\sum_{j \in J_0} \mu_j (\pi_j(x) + x^T w_j) \leq 0,\]
using the inequalities and the dual constraint (3.2), hypothesis (i) gives
\[
\eta^T(x, y) \left\{ \nabla \left( \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) + \sum_{j \in J_0} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y, p)\} \right\} < -\rho^1 d^2(x, y)
\]
and
\[
\eta^T(x, y) \sum_{j \in J_0} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y, p)\} + \rho^2 d^2(x, y) \leq 0, \quad \beta = 1, \ldots, r.
\]
Since \(\lambda \geq 0, \lambda^T e = 1\), it follows that
\[
\eta^T(x, y) \left( \sum_{i=1}^k \lambda_i \left\{ \nabla \left( \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right)
+ \sum_{j \in J_0} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y, p)\} < -\sum_{i=1}^k \lambda_i \rho^1 d^2(x, y)
\]
and
\[
\eta^T(x, y) \left( \sum_{j \in J_0} \mu_j \{\nabla \pi_j(y) + w_j + \nabla_p K_j(y, p)\} \right) \leq -\rho^2 d^2(x, y), \quad \beta = 1, \ldots, r.
\]
Above inequalities follows that

\[ \eta^T(x, y) \left( \sum_{i=1}^{k} \lambda_i \left\{ \nabla \left( \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right) + \sum_{j=1}^{m} \mu_j \{ \nabla \pi_j(y) + w_j \}
\]

\[ + \nabla_p K_j(y, p) \right) = \eta^T(x, y) \left( \sum_{i=1}^{k} \lambda_i \left\{ \nabla \left( \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right) 
\]

\[ + \sum_{j \in J_0} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} + \sum_{j \in J_1} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \}
\]

\[ + \ldots + \sum_{j \in J_r} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} \]

\[ \eta^T(x, y) \left( \sum_{i=1}^{k} \lambda_i \left\{ \nabla \left( \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right) + \sum_{j=1}^{m} \mu_j \{ \nabla \pi_j(y) + w_j \}
\]

\[ + \nabla_p K_j(y, p) \right) \leq \eta^T(x, y) \left( \sum_{i=1}^{k} \lambda_i \left\{ \nabla \left( \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right) 
\]

\[ + \sum_{j \in J_0} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} + \sum_{j \in J_1} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} 
\]

\[ + \nabla_p K_j(y, p) \right) \ldots + \sum_{j \in J_r} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} \]

\[ < - \left( \sum_{i=1}^{k} \lambda_i \rho_i^1 + \sum_{j=1}^{r} \mu_j \rho_j^2 \right) d^2(x, y). \]

Further, using hypothesis (ii), we have

\[ \eta^T(x, y) \left( \sum_{i=1}^{k} \lambda_i \left\{ \nabla \left( \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} \right) + \nabla_p H_i(y, p) \right\} \right) 
\]

\[ + \sum_{j=1}^{m} \mu_j \{ \nabla \pi_j(y) + w_j + \nabla_p K_j(y, p) \} < 0, \]

which contradicts (3.1). Hence, completes the proof.
**Theorem 3.2 (Weak Duality Theorem).** Let $x \in Y^0$ and $(y, \lambda, v, \mu, z, w, p) \in W^0$. Let

\[
(i) \left( \frac{\phi_i(.) + (.)^T z_i}{\psi_i(.) + (.)^T v_i} + \mu_{j_0} \left( \pi_{j_0} + (.)^T w_{j_0} \right) e, \{\pi_j(.) + (.)^T w_j\}_{j_0}^\mu \right) \text{ be higher-order pseudo quasi (V, p, d)-type I at } y,
\]

\[
(ii) \sum_{i=1}^k \lambda_i \rho_i^1 + \sum_{j=1}^r \mu_j \rho_j^2 \geq 0.
\]

Then, the following cannot hold

\[
\frac{\phi_i(x) + s(x|C_i)}{\psi_i(x) - s(x|E_i)} \leq \frac{\phi_i(y) + y^T z_i}{\psi_i(y) - y^T v_i} + H_i(y, p) - p^T \nabla_p H_i(y, p)
\]

\[+(\sum_{j \in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p)\}, \forall i \in K \tag{3.7}\]

and

\[
\frac{\phi_j(x) + s(x|C_j)}{\psi_j(x) - s(x|E_j)} < \frac{\phi_j(y) + y^T z_j}{\psi_j(x) - y^T v_j} + H_j(y, p) - p^T \nabla_p H_j(y, p)
\]

\[+(\sum_{j \in J_0} \mu_j \{\pi_j(y) + y^T w_j + K_j(y, p) - p^T \nabla_p K_j(y, p)\}, \text{for some } j \in K. \tag{3.8}\]

**Proof** The proof follows on the lines of Theorem 3.1.

**Theorem 3.3 (Strong Duality Theorem).** If $\bar{u}$ is an efficient solution of (MFP) and let the Kuhn-Tucker constraint qualification be satisfied. Then, $\exists \lambda \in R^k, \bar{y} \in R^m, \bar{z} \in R^n, \bar{v}_i \in R^m \text{ and } \bar{w}_j \in R^m, i \in K, j \in M$, such that $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \lambda, \bar{w}, \bar{p}) \in W^0$ and the (MFP) and (HMDP) have equal values. Also, if

\[H(\bar{u}, 0) = 0, \nabla_p H(\bar{u}, 0) = 0, \text{ and } K(\bar{u}, 0) = 0, \nabla_p K(\bar{u}, 0) = 0.\]
Furthermore, if the assumptions of Theorem 3.1 or 3.2 hold for $Y^0$ and $W^0$, then $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0)$ is an efficient solution of (HMDP).

**Proof.** Since $\bar{u}$ is an efficient solution for (MFP) and the Slater’s constraint qualification is satisfied, from Theorem 2.1, there exist $0 < \bar{\lambda} \in R^k$, $0 \leq \bar{y}_j \in R^m$, $\bar{z}_i \in R^n$, $\bar{v}_i$, $\bar{w}_j \in R^n$, $i \in K$, $j \in M$ such that

\begin{equation}
\sum_{i=1}^{k} \bar{\lambda}_i \nabla \left( \phi_i(u) + u^T \bar{z}_i \right) + \sum_{j=1}^{m} \bar{y}_j \nabla (\pi_j(u) + u^T \bar{w}_j) = 0,
\end{equation}

\begin{equation}
\sum_{j=1}^{m} \bar{y}_j (\pi_j(u) + u^T \bar{w}_j) = 0,
\end{equation}

\begin{equation}
u^T \bar{z}_i = S(u|C_i), \quad \nu^T \bar{v}_i = S(u|E_i), \quad \nu^T \bar{w}_j = S(u|D_j),
\end{equation}

\begin{equation}
\bar{z}_i \in C_i, \quad \bar{v}_i \in D_i, \quad \bar{w}_j \in E_j, \quad i \in K, \quad j \in M.
\end{equation}

Using the assumption $H(\bar{u}, 0) = 0$, $\nabla_p H(\bar{u}, 0) = 0$, $K(\bar{u}, 0) = 0$, $\nabla_p K(\bar{u}, 0) = 0$, we find that $(\bar{u}, \bar{z}, \bar{v}, \bar{y}, \bar{\lambda}, \bar{w}, \bar{p} = 0) \in W^0$ and the two objective values are same. With the help of contradiction, we can prove efficiency results. Hence, the results. □

**Theorem 3.4 (Strict Converse Duality Theorem).** Let $u \in Y^0$ and $(\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{v}, \bar{z}, \bar{w}, \bar{p}) \in W^0$ such that

\begin{enumerate}[(i)]
\item \[
\sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right\} \leq \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z}_i}{\psi_i(\bar{y}) - \bar{y}^T \bar{v}_i} + \nabla_p H_i(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p H_i(\bar{y}, \bar{p}) \right\} + \sum_{j \in J_0} \bar{\mu}_j \left\{ \pi_j(\bar{y}) + \bar{y}^T \bar{w}_j + K_j(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p K_j(\bar{y}, \bar{p}) \right\},
\end{enumerate}
(ii) \[ \rho_1^i + \sum_{j=1}^{r} \rho_2^j \geq 0, \forall \ i, \ j, \]

(iii) \[ \left( \sum_{i=1}^{k} \bar{\lambda}_i \left( \frac{\phi_i(.) + (.)^T \bar{z}_i}{\psi_i(.) - (.)^T \bar{v}_i} \right) + \sum_{j \in J_0} \mu_j \left\{ \pi_j + (.)^T \bar{w}_j \right\}, \left\{ \pi_j(.) + (.)^T \bar{w}_j \right\} \right) \]

is higher order strictly pseudoquasi \((V, \rho, \bar{\beta}) - type Iat \bar{y} \).

Then, \( u = \bar{y} \).

**Proof.** Suppose \( u \neq \bar{y} \). The dual constraint (3.2) and the hypothesis (iii), for \( \beta = 1, ..., r \) yield

\[ (3.13) \eta^T(u, \bar{y}) \sum_{j \in J_\beta} \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) + \nabla \rho K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \leq -\rho_2^2 \bar{d}^2(u, \bar{y}) \]

By the dual constraint (3.1), we have

\[ \eta^T(u, \bar{y}) \left( \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \nabla \left( \frac{\phi_i(u) + (u)^T \bar{z}_i}{\psi_i(u) - (u)^T \bar{v}_i} \right) + \nabla H_i(\bar{y}, \bar{p}) \right\} \right) \]

\[ + \sum_{j \in J_0} \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) + \nabla \rho K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \right) = 0, \]

above inequalities with (3.13) give

\[ \eta^T(u, \bar{y}) \left( \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \nabla \left( \frac{\phi_i(u) + (u)^T \bar{z}_i}{\psi_i(u) - (u)^T \bar{v}_i} \right) + \nabla H_i(\bar{y}, \bar{p}) \right\} + \sum_{j \in J_0} \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) + \nabla \rho K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \right) \]

\[ + \nabla \rho K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \right) \geq -\eta^T(u, \bar{y}) \left( \sum_{j \in J_\beta} \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) + \nabla \rho K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \right) \]

\[ -... - \eta^T(u, \bar{y}) \left( \sum_{j \in J_r} \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) + \nabla \rho K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \right) \]

or
by hypothesis (ii),

\[
\eta^T(u, \bar{y}) \left( \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \nabla \left( \frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right) + \nabla_p H_i(\bar{y}, \bar{p}) \right\} \right)
+ \sum_{j \in J_0} \bar{\mu}_j \left\{ \nabla \pi_j(\bar{y}) + \nabla_p K_j(\bar{y}, \bar{p}) + \bar{w}_j \right\} \geq \sum_{j=1}^{r} \rho_j^2 d_j^2 (u, \bar{y}),
\]

which contradicts hypothesis (i). Hence, the result.

**Theorem 3.5 (Strict Converse Duality Theorem).** Let \( u \in Y^0 \) and \( (\bar{y}, \bar{\lambda}, \bar{\mu}, \bar{v}, \bar{z}, \bar{w}, \bar{p}) \in W^0 \) such that

1. \( \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(u) + u^T \bar{z}_i}{\psi_i(u) - u^T \bar{v}_i} \right\} \leq \sum_{i=1}^{k} \bar{\lambda}_i \left\{ \frac{\phi_i(\bar{y}) + \bar{y}^T \bar{z}_i}{\psi_i(\bar{y}) - \bar{y}^T \bar{v}_i} + \nabla_p H_i(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p H_i(\bar{y}, \bar{p}) \right\}
+ \sum_{j \in J_0} \bar{\mu}_j \left\{ \pi_j(\bar{y}) + \bar{y}^T \bar{w}_j + K_j(\bar{y}, \bar{p}) - \bar{p}^T \nabla_p K_j(\bar{y}, \bar{p}) \right\},
\]

2. \( \rho_i^1 + \sum_{j=1}^{r} \rho_j^2 \geq 0, \ \forall \ i, \ j, \)
\[(iii) \quad \left( \sum_{i=1}^{k} \lambda_i \left\{ \phi_i(.,+) + \left(\cdot^{T} e_i \right) \right\} + \sum_{j \in J_0} \mu_j \left\{ \pi_j + \left(\cdot^{T} w_j \right) \right\}, \{ \pi_j(.,+) + \left(\cdot^{T} w_j \right) \}_J \right) \]

is higher order quasistrictly pseudo \((V, \rho, d) - type\) at \(\tilde{y}\).

Then, \(u = \tilde{y}\).

**Proof** The proof follows on the lines of theorem 3.4.

References


