Non-linear new product $A^*B - B^*A$ derivations on $\ast$-algebras

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Abstract:

Let $A$ be a prime $\ast$-algebra with unit $1$ and a nontrivial projection. Then the map $\Phi : A \to A$ satisfies in the following condition

$$\Phi(A \ast B) = \Phi(A) \ast B + A \ast \Phi(B)$$

where $A \ast B = A^*B - B^*A$ for all $A, B \in A$, is additive. Moreover, if $\Phi(\alpha 1)$ is self-adjoint operator for $\alpha \in \{1, i\}$ then $\Phi$ is a $\ast$-derivation.

Keywords: New product derivation; Prime $\ast$-algebra; Additive map.

1. Introduction

Let $\mathcal{R}$ be a $*$-algebra. For $A, B \in \mathcal{R}$, denoted by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are $*$-Jordan product and $*$-Lie product, respectively. These products are found playing a more and more important role in some research topics, and its study has recently attracted many author’s attention (for example, see [3, 9, 11, 15]).

Recall that a map $\Phi : \mathcal{R} \to \mathcal{R}$ is said to be an additive derivation if

$$
\Phi(A + B) = \Phi(A) + \Phi(B)
$$

and

$$
\Phi(AB) = \Phi(A)B + A\Phi(B)
$$

for all $A, B \in \mathcal{R}$. A map $\Phi$ is additive $*$-derivation if it is an additive derivation and $\Phi(A^*) = \Phi(A)^*$. Derivations are very important maps both in theory and applications, and have been studied intensively ([2, 12, 13, 14]).

Let us define $\lambda$-Jordan $*$-product by $A \bullet_\lambda B = AB + \lambda BA^*$. We say that the map $\Phi$ with the property of $\Phi(A \bullet_\lambda B) = \Phi(A) \bullet_\lambda B + A \bullet_\lambda \Phi(B)$ is a $\lambda$-Jordan $*$-derivation map. It is clear that for $\lambda = -1$ and $\lambda = 1$, the $\lambda$-Jordan $*$-derivation map is a $*$-Lie derivation and $*$-Jordan derivation, respectively [1].

A von Neumann algebra $A$ is a self-adjoint subalgebra of some $B(H)$, the algebra of bounded linear operators acting on a complex Hilbert space, which satisfies the double commutant property: $A'' = A$ where $A' = \{T \in B(H) : TA = AT, \forall A \in A\}$ and $A'' = \{A'\}'$. Denote by $Z(A) = A' \cap A$ the center of $A$. A von Neumann algebra $A$ is called a factor if its center is trivial, that is, $Z(A) = CI$. For $A \in A$, recall that the central carrier of $A$, denoted by $\mathcal{A}$, is the smallest central projection $P$ such that $PA = A$. It is not difficult to see that $\mathcal{A}$ is the projection onto the closed subspace spanned by $\{B Ax : B \in A, x \in H\}$. If $A$ is self-adjoint, then the core of $A$, denoted by $\mathcal{A}$, is sup$\{S \in Z(A) : S = S^*, S \leq A\}$. If $A = P$ is a projection, it is clear that $P$ is the largest central projection $Q$ satisfying $Q \leq P$. A projection $P$ is said to be core-free if $P = 0$ (see [10]). It is easy to see that $P = 0$ if and only if $I - P = I$, [6, 7].

Recently, Yu and Zhang in [17] proved that every non-linear $*$-Lie derivation from a factor von Neumann algebra into itself is an additive $*$-derivation. Also, Li, Lu and Fang in [8] have investigated a non-linear $\lambda$-Jordan $*$-derivation. They showed that if $A \subseteq B(H)$ is a von Neumann algebra without central abelian projections and $\lambda$ is a non-zero scalar, then
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$\Phi : A \rightarrow B(\mathcal{H})$ is a non-linear $\lambda$-Jordan $*$-derivation if and only if $\Phi$ is an additive $*$-derivation.

On the other hand, many mathematician devoted themselves to study the $*$-Jordan product $A \bullet B = AB + BA^*$. In [18], F. Zhang proved that every non-linear $*$-Jordan derivation map $\Phi : A \rightarrow A$ on a factor von neumann algebra with $I_A$ the identity of it is an additive $*$-derivation.

In [16], we showed that $*$-Jordan derivation map on every factor von Neumann algebra $A \subseteq B(\mathcal{H})$ is additive $*$-derivation.

Very recently the authors of [5] discussed some bijective maps preserving the new product $A^*B + B^*A$ between von Neumann algebras with no central abelian projections. In other words, $\Phi$ holds in the following condition

$$\Phi(A^*B + B^*A) = \Phi(A)^*\Phi(B) + \Phi(B)^*\Phi(A).$$

They showed that such a map is sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism.

Motivated by the above results, in this paper, we prove that if $A$ is a prime $*$-algebra then $\Phi : A \rightarrow A$ which holds in the following condition

$$\Phi(A \circ B) = \Phi(A) \circ B + A \circ \Phi(B)$$

where $A \circ B = A^*B - B^*A$ for all $A, B \in A$, is additive $*$-derivation.

We say that $A$ is prime, that is, for $A, B \in A$ if $AAB = \{0\}$, then $A = 0$ or $B = 0$. For example, every simple or prime generally primitive $C^*$-algebras are prime (e.g., $B(\mathcal{H}), K(\mathcal{H})$ for every Hilbert space) [4].

2. Main Results

Our main theorem is as follows:

**Theorem 2.1.** Let $A$ be a prime $*$-algebra with unit $I$ and a nontrivial projection. Then the map $\Phi : A \rightarrow A$ satisfies in the following condition

$$\Phi(A \circ B) = \Phi(A) \circ B + A \circ \Phi(B)$$

where $A \circ B = A^*B - B^*A$ for all $A, B \in A$. is additive.

**Proof.** Let $P_1$ be a nontrivial projection in $A$ and $P_2 = I_A - P_1$. Denote $A_{ij} = P_iAP_j, \ i, j = 1, 2$, then $A = \sum_{i,j=1}^2 A_{ij}$. For every $A \in A$ we may write $A = A_{11} + A_{12} + A_{21} + A_{22}$. In all that follow, when we write $A_{ij}$, it indicates that $A_{ij} \in A_{ij}$. For showing additivity of $\Phi$ on $A$, we use
above partition of $\mathcal{A}$ and give some claims that prove $\Phi$ is additive on each $\mathcal{A}_{ij}$, $i, j = 1, 2$.

We prove the above theorem by several claims.

**Claim 1.** $\Phi(0) = 0$.

This claim is easy to prove.

**Claim 2.** $\Phi(iA) = i\Phi(A) + A^*K$, where, $K = \Phi(iI) - i\Phi(I)$.

Consider

\[ \Phi(-iA \circ I) = \Phi(A \circ iI). \]

So, we have

\[ \Phi(-iA)^* - \Phi(-iA) + iA^*\Phi(I) + i\Phi(I)^*A = i\Phi(A)^* + i\Phi(A) \]

\[ + A^*\Phi(iI) - \Phi(iI)^*A. \]

(2.2)

Consider

\[ \Phi(-iA \circ iI) = \Phi(I \circ A) \]

So, we have

\[ i\Phi(-iA)^* + i\Phi(-iA) + iA^*\Phi(iI) + i\Phi(iI)^*A = \Phi(I)^*A - A^*\Phi(I) + \Phi(A) - \Phi(A)^*. \]

Equivalently, we obtain

\[ -\Phi(-iA)^* - \Phi(-iA) - A^*\Phi(iI) - \Phi(iI)^*A = i\Phi(I)^*A - iA^*\Phi(I) \]

\[ + i\Phi(A) - i\Phi(A)^*. \]

(2.3)

By adding equations (2.2) and (2.3) we have

\[ -\Phi(-iA) - i\Phi(A) = -iA^*\Phi(I) + A^*\Phi(iI). \]

Substituting $iA$ instead of $A$ in the above equation implies

\[ \Phi(iA) = i\Phi(A) + A^*(\Phi(iI) - i\Phi(I)) \]

that

\[ K = \Phi(iI) - i\Phi(I). \]

So

\[ \Phi(iA) = i\Phi(A) + A^*K \]
Claim 3. \( \Phi(-A) = -\Phi(A) \)

By considering \( \Phi(iA) = i\Phi(A) + A^*K \) and applying \( iA \) instead of \( A \) we have

\[
\begin{align*}
\Phi(-A) &= i\Phi(iA) - iA^*K \\
&= i(i\Phi(A) + A^*K) - iA^*K \\
&= -\Phi(A) + iA^*K - iA^*K \\
&= -\Phi(A) \\
\end{align*}
\]

(2.4)

Claim 4. For each \( A_{11} \in A_{11}, \ A_{12} \in A_{12} \) we have

\[
\Phi(A_{11} + A_{12}) = \Phi(A_{11}) + \Phi(A_{12}).
\]

Let \( T = \Phi(A_{11} + A_{12}) - \Phi(A_{11}) - \Phi(A_{12}) \), we should prove that \( T = 0 \).

For \( X_{21} \in A_{21} \) we can write that

\[
\begin{align*}
\Phi(A_{11} + A_{12}) &\circ X_{21} + (A_{11} + A_{12}) \circ \Phi(X_{21}) = \Phi((A_{11} + A_{12}) \circ X_{21}) \\
&= \Phi(A_{11} \circ X_{21}) + \Phi(A_{12} \circ X_{21}) = \Phi(A_{11}) \circ X_{21} + A_{11} \circ \Phi(X_{21}) \\
&\quad + \Phi(A_{12}) \circ X_{21} + A_{12} \circ \Phi(X_{21}) \\
&= (\Phi(A_{11}) + \Phi(A_{12})) \circ X_{21} + (A_{11} + A_{12}) \circ \Phi(X_{21}).
\end{align*}
\]

So, we obtain

\[
T \circ X_{21} = 0.
\]

Since \( T = T_{11} + T_{12} + T_{21} + T_{22} \) we have

\[
T_{21}^*X_{21} + T_{22}^*X_{21} - X_{21}^*T_{21} - X_{21}^*T_{22} = 0.
\]

From the above equation and primeness of \( A \) we have \( T_{22} = 0 \) and

(2.5) \( T_{21}^*X_{21} - X_{21}^*T_{21} = 0. \)

On the other hand, similarly by applying \( iX_{21} \) instead of \( X_{21} \) in above, we obtain

\[
iT_{21}^*X_{21} + iT_{22}^*X_{21} + iX_{21}^*T_{21} + iX_{21}^*T_{22} = 0.
\]

Since \( T_{22} = 0 \) we obtain from the above equation that

(2.6) \( -T_{21}^*X_{21} - X_{21}^*T_{21} = 0. \)

From (2.5) and (2.6) we have

\[
X_{21}^*T_{21} = 0.
\]
Since \( A \) is prime, then we get \( T_{21} = 0 \).

It suffices to show that \( T_{12} = T_{11} = 0 \). For this purpose for \( X_{12} \in A_{12} \) we write

\[
\Phi(((A_{11} + A_{12}) \circ X_{12}) \circ P_1) = \Phi((A_{11} + A_{12}) \circ X_{12}) \circ P_1 + ((A_{11} + A_{12}) \circ X_{12}) \circ \Phi(P_1)
\]

\[
= (\Phi(A_{11} + A_{12}) \circ X_{12} + (A_{11} + A_{12}) \circ \Phi(X_{12})) \circ P_1 + (A_{11} + A_{12}) \circ X_{12} \circ \Phi(P_1)
\]

\[
+ A_{11} \circ X_{12} \circ \Phi(P_1) + A_{12} \circ X_{12} \circ \Phi(P_1).
\]

So, we showed that

\[
\Phi(((A_{11} + A_{12}) \circ X_{12}) \circ P_1) = \Phi(A_{11} + A_{12}) \circ X_{12} \circ P_1 + A_{11} \circ \Phi(X_{12}) \circ P_1
\]

\[
+ A_{12} \circ \Phi(X_{12}) \circ P_1 + A_{11} \circ X_{12} \circ \Phi(P_1) + A_{12} \circ X_{12} \circ \Phi(P_1).
\]

(2.7)

Since \( A_{12} \circ X_{12} \circ P_1 = 0 \) we have

\[
\Phi(((A_{11} + A_{12}) \circ X_{12}) \circ P_1) = \Phi((A_{11} \circ X_{12}) \circ P_1) + \Phi((A_{12} \circ X_{12}) \circ P_1)
\]

\[
= \Phi(A_{11} \circ X_{12}) \circ P_1 + (A_{11} \circ X_{12}) \circ \Phi(P_1) + \Phi(A_{12} \circ X_{12}) \circ P_1 + (A_{12} \circ X_{12}) \circ \Phi(P_1)
\]

\[
= (\Phi(A_{11}) \circ X_{12} + A_{11} \circ \Phi(X_{12})) \circ P_1 + (A_{11} \circ X_{12}) \circ \Phi(P_1)
\]

\[
+ (\Phi(A_{12}) \circ X_{12} + A_{12} \circ \Phi(X_{12})) \circ P_1 + (A_{12} \circ X_{12}) \circ \Phi(P_1)
\]

\[
= \Phi(A_{11}) \circ X_{12} \circ P_1 + A_{11} \circ \Phi(X_{12}) \circ P_1 + A_{11} \circ X_{12} \circ \Phi(P_1)
\]

\[
+ \Phi(A_{12}) \circ X_{12} \circ P_1 + A_{12} \circ \Phi(X_{12}) \circ P_1 + A_{12} \circ X_{12} \circ \Phi(P_1).
\]

So,

\[
\Phi(((A_{11} + A_{12}) \circ X_{12}) \circ P_1) = \Phi(A_{11}) \circ X_{12} \circ P_1 + A_{11} \circ \Phi(X_{12}) \circ P_1
\]

\[
+ A_{12} \circ \Phi(X_{12}) \circ P_1 + A_{11} \circ X_{12} \circ \Phi(P_1)
\]

(2.8) +\( A_{12} \circ \Phi(X_{12}) \circ P_1 + A_{11} \circ X_{12} \circ \Phi(P_1).

From (2) and (2.7) we have

\[
\Phi(A_{11} + A_{12}) \circ X_{12} \circ P_1 = \Phi(A_{11}) \circ X_{12} \circ P_1 + \Phi(A_{12}) \circ X_{12} \circ P_1.
\]

It follows that \( T \circ X_{12} \circ P_1 = 0 \), so \( T_{11}^* X_{12} - X_{12}^* T_{11} = 0 \). We have \( T_{11}^* X_{12} = 0 \) or \( T_{11} X P_2 = 0 \) for all \( X \in A \), then we have \( T_{11} = 0 \). Similarly, we can show that \( T_{12} = 0 \) by applying \( P_2 \) instead of \( P_1 \) in above.

**Claim 5.** For each \( A_{11} \in A_{11}, A_{12} \in A_{12}, A_{21} \in A_{21} \) and \( A_{22} \in A_{22} \) we have
1. 
\[ \Phi(A_{11} + A_{12} + A_{21}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}). \]

2. 
\[ \Phi(A_{12} + A_{21} + A_{22}) = \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}). \]

We show that 
\[ T = \Phi(A_{11} + A_{12} + A_{21}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) = 0. \]

So, we have 
\[ \Phi(A_{11} + A_{12} + A_{21}) \odot X_{21} + (A_{11} + A_{12} + A_{21}) \odot \Phi(X_{21}) \]
\[ = \Phi((A_{11} + A_{12} + A_{21}) \odot X_{21}) = \Phi(A_{11} \odot X_{21}) + \Phi(A_{12} \odot X_{21}) + \Phi(A_{21} \odot X_{21}) \]
\[ = (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \odot X_{21} + (A_{11} + A_{12} + A_{21}) \odot \Phi(X_{21}). \]

It follows that \( T \odot X_{21} = 0 \). Since \( T = T_{11} + T_{12} + T_{21} + T_{22} \) we have 
\[ T_{22}^*X_{21} + T_{21}^*X_{21} - X_{21}^*T_{22} - C_{21}^*T_{21} = 0. \]

Therefore, \( T_{22} = T_{21} = 0 \).

From Claim 4, we obtain 
\[ \Phi(A_{11} + A_{12} + A_{21}) \odot X_{12} + (A_{11} + A_{12} + A_{21}) \odot \Phi(X_{12}) \]
\[ = \Phi((A_{11} + A_{12} + A_{21}) \odot X_{12}) = \Phi((A_{11} + A_{12}) \odot X_{12}) + \Phi(A_{21} \odot X_{12}) \]
\[ = \Phi(A_{11} \odot X_{12}) + \Phi(A_{12} \odot X_{12}) + \Phi(A_{21} \odot X_{12}) \]
\[ = (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21})) \odot X_{12} + (A_{11} + A_{12} + A_{21}) \odot \Phi(X_{12}). \]

Hence, 
\[ T_{11}^*X_{12} + T_{12}^*X_{12} - X_{12}^*T_{11} - X_{12}^*T_{12} = 0. \]

Then \( T_{11} = T_{12} = 0 \). Similarly 
\[ \Phi(A_{12} + A_{21} + A_{22}) = \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}). \]

**Claim 6.** For each \( A_{11} \in A_{11}, A_{12} \in A_{12}, A_{21} \in A_{21} \) and \( A_{22} \in A_{22} \) we have 
\[ \Phi(A_{11} + A_{12} + A_{21} + A_{22}) = \Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22}). \]
We show that
\[ T = \Phi(A_{11} + A_{12} + A_{21} + A_{22}) - \Phi(A_{11}) - \Phi(A_{12}) - \Phi(A_{21}) - \Phi(A_{22}) = 0. \]

From Claim 5, we have
\[
\Phi(A_{11} + A_{12} + A_{21} + A_{22}) \circ X_{12} + (A_{11} + A_{12} + A_{21} + A_{22}) \circ \Phi(X_{12})
\]
\[
= \Phi((A_{11} + A_{12} + A_{21} + A_{22}) \circ X_{12})
\]
\[
= \Phi((A_{11} \circ X_{12}) + (A_{12} \circ X_{12}) + (A_{21} \circ X_{12}) + (A_{22} \circ X_{12}))
\]
\[
= (\Phi(A_{11}) + \Phi(A_{12}) + \Phi(A_{21}) + \Phi(A_{22})) \circ X_{12}
\]
\[+ (A_{11} + A_{12} + A_{21} + A_{22}) \circ \Phi(X_{12}). \]

So, \( T \circ X_{12} = 0 \). It follows that
\[ T_{11}^* X_{12} + T_{12}^* X_{12} - X_{12}^* T_{11} - X_{12}^* T_{12} = 0. \]

Then \( T_{11} = T_{12} = 0 \).

Similarly, by applying \( X_{21} \) instead of \( X_{12} \) in above, we obtain \( T_{21} = T_{22} = 0 \).

**Claim 7.** For each \( A_{ij}, B_{ij} \in A_{ij} \) such that \( i \neq j \), we have
\[ \Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}). \]

It is easy to show that
\[ (P_i + A_{ij})(P_j + B_{ij}) - (P_j + B_{ij})(P_i + A_{ij}^*) = A_{ij} + B_{ij} - A_{ij}^* - B_{ij}^*. \]

So, we can write
\[
\Phi(A_{ij} + B_{ij}) + \Phi(-A_{ij}^* - B_{ij}^*) = \Phi((P_i + A_{ij}^*) \circ (P_j + B_{ij}))
\]
\[
= \Phi(P_i + A_{ij}^*) \circ (P_j + B_{ij}) + (P_i + A_{ij}^*) \circ \Phi(P_j + B_{ij})
\]
\[
= (\Phi(P_i) + \Phi(A_{ij}^*)) \circ (P_j + B_{ij}) + (P_i + A_{ij}^*) \circ (\Phi(P_j) + \Phi(B_{ij}))
\]
\[
= \Phi(P_i) \circ B_{ij} + P_i \circ \Phi(B_{ij}) + \Phi(A_{ij}^*) \circ P_j + A_{ij}^* \circ \Phi(P_j)
\]
\[
= \Phi(P_i \circ B_{ij}) + \Phi(A_{ij}^* \circ P_j)
\]
\[
= \Phi(B_{ij}) + \Phi(-B_{ij}^*) + \Phi(A_{ij}) + \Phi(-A_{ij}^*). \]

Therefore, we show that
(2.9) \( \Phi(A_{ij} + B_{ij}) + \Phi(-A_{ij}^* - B_{ij}^*) = \Phi(A_{ij}) + \Phi(B_{ij}) + \Phi(-A_{ij}^*) + \Phi(-B_{ij}^*) \)

By an easy computation, we can write

\[
(P_1 + A_{ij})(iP_j + iB_{ij}) - (-iP_j - iB_{ij}^*)(P_1 + A_{ij}^*) = iA_{ij} + iB_{ij} + iA_{ij}^* + iB_{ij}^*.
\]

Then, we have

\[
\Phi(iA_{ij} + iB_{ij}) + \Phi(iA_{ij}^* + iB_{ij}^*) = \Phi((P_1 + A_{ij}^*) \circ (iP_j + iB_{ij}))
\]

\[
= \Phi(P_1 + A_{ij}^*) \circ (iP_j + iB_{ij}) + (P_1 + A_{ij}^*) \circ \Phi(iP_j + iB_{ij})
\]

\[
= (\Phi(P_1) + \Phi(A_{ij}^*)) \circ (iP_j + iB_{ij}) + (P_1 + A_{ij}^*) \circ \Phi(iP_j + iB_{ij})
\]

\[
= \Phi(P_1) \circ iP_j + P_1 \circ \Phi(iP_j) + \Phi(A_{ij}^*) \circ iP_j + A_{ij}^* \circ \Phi(iP_j)
\]

\[
= \Phi(P_1 \circ iP_j) + \Phi(A_{ij}^* \circ iP_j)
\]

\[
= \Phi(iB_{ij}^*) + \Phi(iB_{ij}) + \Phi(iA_{ij}^*) + \Phi(iA_{ij}).
\]

We showed that

\[
\Phi(iA_{ij} + iB_{ij}) + \Phi(iA_{ij}^* + iB_{ij}^*) = \Phi(iB_{ij}) + \Phi(iB_{ij}^*) + \Phi(iA_{ij}) + \Phi(iA_{ij}^*).
\]

From Claims 2, 3 and the above equation, we have

(2.10) \( \Phi(A_{ij} + B_{ij}) - \Phi(-A_{ij}^* - B_{ij}^*) = \Phi(B_{ij}) - \Phi(-B_{ij}^*) + \Phi(A_{ij}) - \Phi(-A_{ij}^*) \)

By adding equations (2.8) and (2.9), we obtain

\[
\Phi(A_{ij} + B_{ij}) = \Phi(A_{ij}) + \Phi(B_{ij}).
\]

Claim 8. For each \( A_{ii}, B_{ii} \in A_{ii} \) such that \( 1 \leq i \leq 2 \), we have

\[
\Phi(A_{ii} + B_{ii}) = \Phi(A_{ii}) + \Phi(B_{ii}).
\]

We show that

\[
T = \Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii}) = 0.
\]

We can write

\[
\Phi(A_{ii} + B_{ii}) \circ P_j + (A_{ii} + B_{ii}) \circ \Phi(P_j) = \Phi((A_{ii} + B_{ii}) \circ P_j)
\]

\[
= \Phi(A_{ii} \circ P_j) + \Phi(B_{ii} \circ P_j)
\]

\[
\Phi(A_{ii}) \circ P_j + A_{ii} \circ \Phi(P_j) + \Phi(B_{ii}) \circ P_j + B_{ii} \circ \Phi(P_j)
\]

\[
= (\Phi(A_{ii}) + \Phi(B_{ii})) \circ P_j + (A_{ii} + B_{ii}) \circ \Phi(P_j).
\]
So, we have

\[ T \diamond P_j = 0. \]

Therefore, we obtain \( T_{ij} = T_{ji} = T_{jj} = 0. \)

On the other hand, for every \( X_{ij} \in A_{ij}, \) we have

\[
\Phi(A_{ii} + B_{ii}) \diamond X_{ij} + (A_{ii} + B_{ii}) \diamond \Phi(X_{ij}) = \Phi((A_{ii} + B_{ii}) \diamond X_{ij}) \\
= \Phi(A_{ii} \diamond X_{ij}) + \Phi(B_{ii} \diamond X_{ij}) = \Phi(A_{ii}) \diamond X_{ij} + A_{ii} \diamond \Phi(X_{ij}) \\
+ \Phi(B_{ii}) \diamond X_{ij} + B_{ii} \diamond \Phi(X_{ij}) \\
= (\Phi(A_{ii}) + \Phi(B_{ii})) \diamond X_{ij} + (A_{ii} + B_{ii}) \diamond \Phi(X_{ij}).
\]

So,

\[
(\Phi(A_{ii} + B_{ii}) - \Phi(A_{ii}) - \Phi(B_{ii})) \diamond X_{ij} = 0.
\]

It follows that \( T \diamond X_{ij} = 0 \) or \( T_{ii}X_{ij} = 0. \) By knowing that \( A \) is prime, we have \( T_{ii} = 0. \)

Hence, the additivity of \( \Phi \) comes from the above claims.

In the rest of this paper we show that \( \Phi \) is \(*\)-derivation.

**Theorem 2.2.** With notation of the previous theorem, if \( \Phi(\alpha I) \) is self-adjoint operator for \( \alpha \in \{1, i\} \) then \( \Phi \) is \(*\)-derivation.

**Proof.** We present the proof of the above theorem by several claims.

**Claim 9.** \( \Phi(iI) = \Phi(I) = 0. \)

Consider \( \Phi(I \diamond iI) = \Phi(I) \diamond iI + I \diamond \Phi(iI) \) that imply

\[
2\Phi(iI) = i\Phi(I)^* + i\Phi(I) + \Phi(iI) - \Phi(iI)^* = i\Phi(I). 
\]

By taking the adjoint of above equation we have \( \Phi(iI) = \Phi(I) = 0 \)

**Claim 10.** \( \Phi \) preserves star.

Since \( \Phi(I) = 0 \) then we can write

\[
\Phi(I \diamond A) = I \diamond \Phi(A).
\]

Then

\[
\Phi(A - A^*) = \Phi(A) - \Phi(A)^*.
\]

So, we showed that \( \Phi \) preserves star.
Claim 11. We prove that $\Phi$ is derivation.

For every $A, B \in \mathcal{A}$ we have

$$
\Phi(AB - B^*A^*) = \Phi(A^* \circ B)
= \Phi(A^*) \circ B + A^* \circ \Phi(B)
= \Phi(A^*)^* B - \Phi(B)^* A^* - B^* \Phi(A^*) + A \Phi(B).
$$

On the other hand, since $\Phi$ preserves star, we have

$$(2.12) \quad \Phi(AB - B^*A^*) = \Phi(A)B + A \Phi(B) - B^* \Phi(A^*) - \Phi(B)^* A^*.$$  

So, from (2.11), we have

$$
\Phi(i(AB + B^*A^*)) = \Phi(A(iB) - (iB)^*A^*)
= \Phi(A)(iB) + A \Phi(iB) - (iB)^* \Phi(A^*) - \Phi(iB)^* A^*.
$$

Therefore, from claims 2 and 9 we have

$$(2.13) \quad \Phi(AB + B^*A^*) = \Phi(A)B + A \Phi(B) + B^* \Phi(A^*) + \Phi(B^*)A^*.$$  

By adding equations (2.11) and (2.12), we have

$$
\Phi(AB) = \Phi(A)B + A \Phi(B).
$$

This completes the proof.

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References


