Some bounds for relative automcommutativity degree

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Abstract:

We consider the probability that a randomly chosen element of a subgroup of a finite group G is fixed by an automorphism of G. We obtain several bounds for this probability and characterize some finite groups with respect to this probability.

Keywords: Autocommutativity degree; Automorphism group; Autoisoclism.

1. Introduction

Let $G$ be a finite group and $\text{Aut}(G)$ be its automorphism group. The relative autocommutativity degree $\text{Pr}(K, \text{Aut}(G))$ of a subgroup $K$ of $G$ is the probability that a randomly chosen element of $K$ is fixed by an automorphism of $G$. In other words

$$\text{Pr}(K, \text{Aut}(G)) = \frac{|\{(a, \nu) \in K \times \text{Aut}(G) : \nu(a) = a\}|}{|K||\text{Aut}(G)|}.$$  

(1.1)

The notion of $\text{Pr}(K, \text{Aut}(G))$ was introduced in [6] and studied in [6, 10]. A generalization of $\text{Pr}(K, \text{Aut}(G))$ can also be found in [2, 11]. Note that $\text{Pr}(G, \text{Aut}(G))$ is the probability that an automorphism of $G$ fixes an element of it. The ratio $\text{Pr}(G, \text{Aut}(G))$ is also known as the autocommutativity degree of $G$. It is worth mentioning that autocommutativity degree of $G$ was initially studied by Sherman [12] in 1975.

In this paper, we obtain several bounds for $\text{Pr}(K, \text{Aut}(G))$. We remark that some of these bounds are better than some existing bounds. We also characterize some finite groups with respect to $\text{Pr}(K, \text{Aut}(G))$. We shall conclude this paper showing that the bounds for $\text{Pr}(K, \text{Aut}(G))$ are also applicable for $\text{Pr}(K_1, \text{Aut}(G_1))$ if $(K_1, G_1)$ and $(K, G)$ are autoisoclinic.

For any element $a \in G$ and $\nu \in \text{Aut}(G)$ we write $[a, \nu] := a^{-1}\nu(a)$, the automcommutator of $a$ and $\nu$. We also write $S(K, \text{Aut}(G)) := \{[a, \nu] : a \in K \text{ and } \nu \in \text{Aut}(G)\}$, $L(K, \text{Aut}(G)) := \{a \in K : \nu(a) = a \text{ for all } \nu \in \text{Aut}(G)\}$ and $[K, \text{Aut}(G)] := \langle S(K, \text{Aut}(G)) \rangle$. Note that $L(K, \text{Aut}(G))$ is a normal subgroup of $K$ contained in $K \cap Z(G)$ and $L(K, \text{Aut}(G)) = \bigcap_{\nu \in \text{Aut}(G)} C_K(\nu)$, where $Z(G)$ is the center of $G$ and $C_K(\nu) := \{a \in K : \nu(a) = a\}$ is a subgroup of $K$. If $K = G$ then $L(K, \text{Aut}(G)) = L(G)$, the absolute centre of $G$ (see [5]). It is also not difficult to see that $K$ is abelian if $\frac{K}{L(K, \text{Aut}(G))}$ is cyclic. Let $C_{\text{Aut}(G)}(a) := \{\nu \in \text{Aut}(G) : \nu(a) = a\}$ for $a \in K$ and $C_{\text{Aut}(G)}(K) := \{\nu \in \text{Aut}(G) : \nu(a) = a \text{ for all } a \in K\}$. Then $C_{\text{Aut}(G)}(a)$ is a subgroup of $\text{Aut}(G)$ and $C_{\text{Aut}(G)}(K) = \bigcap_{a \in K} C_{\text{Aut}(G)}(a)$.

It is easy to see that

$$\{(a, \nu) \in K \times \text{Aut}(G) : \nu(a) = a\} = \bigsqcup_{a \in K} \{\{a\} \times C_{\text{Aut}(G)}(a)\}$$

$$= \bigsqcup_{\nu \in \text{Aut}(G)} (C_K(\nu) \times \{\nu\}),$$

where $\bigsqcup$ stands for union of disjoint sets. Hence
Some bounds for relative autocommutativity degree

\[ |K||Aut(G)|\Pr(K, Aut(G)) = \sum_{a \in K} |C_{Aut(G)}(a)| = \sum_{\nu \in Aut(G)} |C_K(\nu)|. \]

(1.2)

Also, for \( \nu \in Aut(G) \) and \( a \in G \), \((\nu, a) \mapsto \nu(a)\) is an action of \( Aut(G) \) on \( G \). The orbit of \( a \in G \) is given by \( orb(a) := \{\nu(a) : \nu \in Aut(G)\} \) and \( |orb(a)| = |Aut(G)|/|C_{Aut(G)}(a)| \).

Hence, (1.2) gives the following generalization of [1, Proposition 2]

(1.3)

\[ \Pr(K, Aut(G)) = \frac{1}{|K|} \sum_{a \in K} \frac{1}{|orb(a)|} \]

Note that \( \Pr(K, Aut(G)) = 1 \) if and only if \( K = L(K, Aut(G)) \). Therefore, throughout the paper we consider \( K \neq L(K, Aut(G)) \).

2. Some upper bounds

We begin with the following upper bound for \( \Pr(K, Aut(G)) \).

**Theorem 2.1.** If \( K \) is a subgroup of \( G \) then

\[ \Pr(K, Aut(G)) \leq \frac{1}{2} \left( 1 + \frac{1}{[K : L(K, Aut(G))]} \right) \]

with equality if and only if \( |orb(a)| = 2 \) for all \( a \in K \setminus L(K, Aut(G)) \).

**Proof.** By (1.3), we get

\[ \Pr(K, Aut(G)) = \frac{1}{|K|} \left( |L(K, Aut(G))| + \sum_{a \in K \setminus L(K, Aut(G))} \frac{1}{|orb(a)|} \right). \]

(2.1)

Since \( |orb(a)| \geq 2 \) for all \( a \in K \setminus L(K, Aut(G)) \), the result follows from (2.1).
Corollary 2.2. If $K$ is a non-abelian subgroup of $G$, then $\Pr(K, \text{Aut}(G)) \leq \frac{5}{8}$. Further, $\Pr(K, \text{Aut}(G)) = \frac{5}{8}$ if and only if $|\text{orb}(a)| = 2$ for all $a \in K \setminus L(K, \text{Aut}(G))$ and $\frac{|K|}{|L(K, \text{Aut}(G))|} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Proof. The inequality follows from Theorem 2.1 noting that $|K|/|L(K, \text{Aut}(G))| \geq 4$ if $K$ is non-abelian.

Note that $\Pr(K, \text{Aut}(G)) = \frac{5}{8}$ if and only if $|K|/|L(K, \text{Aut}(G))| = 4$ and equality holds in Theorem 2.1. Hence, the result follows.

Theorem 2.3. If $K$ is a subgroup of $G$ and $p$ the smallest prime dividing $|\text{Aut}(G)|$, then

$$\Pr(K, \text{Aut}(G)) \leq \frac{(p - 1)|L(K, \text{Aut}(G))| + |K|}{p|K|} - \frac{|X_K|(|\text{Aut}(G)| - p)}{p|K||\text{Aut}(G)|}$$

where $X_K = \{a \in K : C_{\text{Aut}(G)}(a) = \{I\}\}$ and $I$ is the identity of $\text{Aut}(G)$.

Proof. Note that $X_K \cap L(K, \text{Aut}(G)) = \emptyset$. Therefore

$$\sum_{a \in K} |C_{\text{Aut}(G)}(a)| = |X_K| + |\text{Aut}(G)||L(K, \text{Aut}(G))| + \sum_{a \in K \setminus (X_K \cup L(K, \text{Aut}(G)))} |C_{\text{Aut}(G)}(a)|.$$

For $a \in K \setminus (X_K \cup L(K, \text{Aut}(G)))$ we have $C_{\text{Aut}(G)}(a) < \text{Aut}(G)$ which implies $|C_{\text{Aut}(G)}(a)| \leq \frac{|\text{Aut}(G)|}{p}$. Therefore

$$\sum_{a \in K} |C_{\text{Aut}(G)}(a)| \leq |X_K| + |\text{Aut}(G)||L(K, \text{Aut}(G))| + \frac{|\text{Aut}(G)||(|K| - |X_K|) - |L(K, \text{Aut}(G))|}{p}.$$

Hence, the result follows from (1.2) and (2.2).

We would like to mention here that Theorem 2.3 gives better upper bound than the upper bound given by [6, Theorem 2.3 (i)]. We also have the following improvement of [6, Corollary 2.2].

Corollary 2.4. Let $K$ be a subgroup of $G$. Then

$$\Pr(K, \text{Aut}(G)) \leq \frac{p + q - 1}{pq}$$

where $p$ and $q$ are the smallest prime divisors of $|\text{Aut}(G)|$ and $|K|$ respectively. Further, if $q \geq p$ then $\Pr(K, \text{Aut}(G)) \leq \frac{2p - 1}{p^2} \leq \frac{3}{4}$.
Proof. We have \( |K : L(K, Aut(G))| \geq q \) since \( K \neq L(K, Aut(G)) \). Therefore, by Theorem 2.3, we get

\[
\Pr(K, Aut(G)) \leq \frac{1}{p} \left( 1 + \frac{p - 1}{|K : L(K, Aut(G))|} \right) \leq \frac{p + q - 1}{pq}.
\]

**Corollary 2.5.** If \( K \) is a non-abelian subgroup of \( G \) then

\[
\Pr(K, Aut(G)) \leq \frac{q^2 + p - 1}{pq^2}
\]

where \( p \) and \( q \) denote respectively the smallest prime divisors of \( |Aut(G)| \) and \( |K| \). Further, if \( q \geq p \) then \( \Pr(K, Aut(G)) \leq \frac{q^2 + p - 1}{pq} \leq \frac{5}{9} \).

Proof. The fact that \( K \) is a non-abelian subgroup of \( G \) implies \( |K : L(K, Aut(G))| \geq q^2 \). Hence

\[
\Pr(K, Aut(G)) \leq \frac{1}{p} \left( 1 + \frac{p - 1}{|K : L(K, Aut(G))|} \right) \leq \frac{q^2 + p - 1}{pq^2}
\]

by Theorem 2.3.

Now we obtain some characterizations of a subgroup \( K \) of \( G \) if equality holds in Corollaries 2.4 and 2.5.

**Theorem 2.6.** If \( K \) is a subgroup of \( G \) and \( \Pr(K, Aut(G)) = \frac{p + q - 1}{pq} \), where \( p, q \) are the smallest prime divisors of \( |Aut(G)| \) and \( |K| \), respectively, then

\[
\frac{K}{L(K, Aut(G))} \cong \mathbb{Z}_q.
\]

**Proof.** If \( p \) and \( q \) denote respectively the smallest prime divisors of \( |Aut(G)| \) and \( |K| \) then, by Theorem 2.3, we get

\[
\frac{p + q - 1}{pq} \leq \frac{1}{p} \left( 1 + \frac{p - 1}{|K : L(K, Aut(G))|} \right)
\]

which gives \( |K : L(K, Aut(G))| \leq q \). Hence, \( \frac{K}{L(K, Aut(G))} \cong \mathbb{Z}_q \).

It is worth mentioning here that Theorem 2.6 generalizes [6, Theorem 2.4].
Theorem 2.7. If $K$ is a subgroup of $G$ and $\Pr(K, \text{Aut}(G)) = \frac{q^2 + p - 1}{pq}$, where $p, q$ are the smallest prime divisors of $|\text{Aut}(G)|$ and $|K|$, respectively, then

$$\frac{K}{L(K, \text{Aut}(G))} \cong \mathbb{Z}_q \times \mathbb{Z}_q.$$  

Further, if $|K|$ and $|\text{Aut}(G)|$ are even and $\Pr(K, \text{Aut}(G)) = \frac{5}{8}$, then

$$\frac{K}{L(K, \text{Aut}(G))} \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$ 

Proof. If $p$ and $q$ denote respectively the smallest prime divisors of $|\text{Aut}(G)|$ and $|K|$, then, by Theorem 2.3, we get

$$\frac{q^2 + p - 1}{pq} \leq \frac{1}{p} \left(1 + \frac{p - 1}{|K : L(K, \text{Aut}(G))|}\right).$$

This gives $|K : L(K, \text{Aut}(G))| \leq q^2$. Since $K$ is non-abelian, $|K : L(K, \text{Aut}(G))| \neq 1, q$. Hence, $\frac{K}{L(K, \text{Aut}(G))} \cong \mathbb{Z}_q \times \mathbb{Z}_q$.

The following result gives partial converses of Theorems 2.6 and 2.7, respectively.

Proposition 2.8. Let $K$ be a subgroup of $G$. Let $p, q$ be the smallest primes dividing $|\text{Aut}(G)|$, $|K|$, respectively, and $|\text{Aut}(G) : C_{\text{Aut}(G)}(a)| = p$ for all $a \in K \setminus L(K, \text{Aut}(G))$.

(a) If $\frac{K}{L(K, \text{Aut}(G))} \cong \mathbb{Z}_q$, then $\Pr(K, \text{Aut}(G)) = \frac{p^2 q - 1}{pq}$.

(b) If $\frac{K}{L(K, \text{Aut}(G))} \cong \mathbb{Z}_q \times \mathbb{Z}_q$, then $\Pr(K, \text{Aut}(G)) = \frac{q^2 + p - 1}{pq}$.

Proof. Since $|\text{Aut}(G) : C_{\text{Aut}(G)}(a)| = p$ for all $a \in K \setminus L(K, \text{Aut}(G))$, we have $|C_{\text{Aut}(G)}(a)| = \frac{|\text{Aut}(G)|}{p}$ for all $a \in K \setminus L(K, \text{Aut}(G))$. Therefore, by (1.2), we get

$$\Pr(K, \text{Aut}(G)) = \frac{|L(K, \text{Aut}(G))|}{|K|} + \frac{1}{|K|} \sum_{a \in K \setminus L(K, \text{Aut}(G))} |C_{\text{Aut}(G)}(a)|$$

Thus

$$\Pr(K, \text{Aut}(G)) = \frac{1}{p} \left(1 + \frac{p - 1}{|K : L(K, \text{Aut}(G))|}\right).$$

Hence, the results follow from (2.3).
For any subgroup $K$ of $G$, let $m_K = \min\{|\text{orb}(a)| : a \in K \setminus L(K, \text{Aut}(G))\}$. The following theorem gives an upper bound for $\text{Pr}(K, \text{Aut}(G))$ involving $m_K$.

**Theorem 2.9.** If $K$ is a subgroup of $G$, then

$$\text{Pr}(K, \text{Aut}(G)) \leq \frac{1}{m_K} \left( 1 + \frac{m_K - 1}{|K : L(K, \text{Aut}(G))|} \right)$$

with equality if and only if $m_K = |\text{orb}(a)|$ for all $a \in K \setminus L(K, \text{Aut}(G))$.

**Proof.** Since $|\text{orb}(a)| \geq m_K$ for all $a \in K \setminus L(K, \text{Aut}(G))$, we have

$$\sum_{a \in K \setminus L(K, \text{Aut}(G))} \frac{1}{|\text{orb}(a)|} \leq \frac{|K| - |L(K, \text{Aut}(G))|}{m_K}.$$

Hence, the result follows from (2.1).

For any two integers $r \geq s$, we have

$$\frac{s}{r} \left( 1 + \frac{s - 1}{|K : L(K, \text{Aut}(G))|} \right) \geq \frac{1}{r} \left( 1 + \frac{r - 1}{|K : L(K, \text{Aut}(G))|} \right).$$

Therefore, if $p$ is the smallest prime dividing $|\text{Aut}(G)|$ then $2 \leq p \leq m_K$ and hence, by (2.4), we have

$$\frac{1}{m_K} \left( 1 + \frac{m_K - 1}{|K : L(K, \text{Aut}(G))|} \right) \leq \frac{1}{p} \left( 1 + \frac{p - 1}{|K : L(K, \text{Aut}(G))|} \right) \leq \frac{1}{2} \left( 1 + \frac{1}{|K : L(K, \text{Aut}(G))|} \right).$$

This shows that Theorem 2.9 gives better upper bound than the upper bounds obtained in [6, Theorem 2.3 (i)] and Theorem 2.1.

Note that if we replace $\text{Aut}(G)$ by the inner automorphism group $\text{Inn}(G)$ of $G$, then from (2.1), we get $\text{Pr}(K, \text{Inn}(G)) = \text{Pr}(K, G)$ where

$$\text{Pr}(K, G) = \frac{|\{(u, v) \in K \times G : uv = vu\}|}{|K||G|}.$$

Various properties of the ratio $\text{Pr}(K, G)$ are studied in [3] and [9]. We conclude this section showing that $\text{Pr}(K, \text{Aut}(G))$ is bounded by $\text{Pr}(K, G)$.

**Proposition 2.10.** If $K$ is a subgroup of $G$ then

$$\text{Pr}(K, \text{Aut}(G)) \leq \text{Pr}(K, G).$$
Proof. From [9, Lemma 1], we get

\[ \Pr(K, G) = \frac{1}{|K|} \sum_{a \in K} \frac{1}{|Cl_G(a)|} \]  

(2.5)

where \( Cl_G(a) = \{ \nu(a) : \nu \in Inn(G) \} \). Since \( Cl_G(a) \subseteq orb(a) \) for all \( a \in K \), the result follows from (1.3) and (2.5).

3. Some lower bounds

We begin this section with the following bound.

Theorem 3.1. If \( K \) a subgroup of \( G \), then

\[ \Pr(K, Aut(G)) \geq \frac{|L(K, Aut(G))|}{|K|} + \frac{p(|K| - |X_K| - |L(K, Aut(G))|)}{|K||Aut(G)|} \]

where \( p \) is the smallest prime dividing \( |Aut(G)| \), \( X_K = \{ a \in K : C_{Aut(G)}(a) = \{ I \} \} \) and \( I \) is the identity of \( Aut(G) \).

Proof. Note that \( X_K \cap L(K, Aut(G)) = \emptyset \). Therefore

\[ \sum_{a \in K} |C_{Aut(G)}(a)| = |X_K| + |Aut(G)||L(K, Aut(G))| + \sum_{a \in K \setminus (X_K \cup L(K, Aut(G)))} |C_{Aut(G)}(a)|. \]

If \( a \in K \setminus (X_K \cup L(K, Aut(G))) \) then \( \{ I \} < C_{Aut(G)}(a) \) which implies \( |C_{Aut(G)}(a)| \geq p. \) Therefore

\[ \sum_{a \in K} |C_{Aut(G)}(a)| \geq |X_K| + |Aut(G)||L(K, Aut(G))| + p(|K| - |X_K| - |L(K, Aut(G))|). \]  

(3.1)

Hence, the result follows from (1.2) and (3.1).

Now we obtain two lower bounds analogous to the lower bounds obtained in [9, Theorem A] and [8, Theorem 1].

Theorem 3.2. If \( K \) is a subgroup of \( G \), then

\[ \Pr(K, Aut(G)) \geq \frac{1}{|S(K, Aut(G))|} \left( 1 + \frac{|S(K, Aut(G))| - 1}{|K : L(K, Aut(G))|} \right) \]

with equality if and only if \( orb(a) = aS(K, Aut(G)) \) for all \( a \in K \setminus L(K, Aut(G)) \).
Proof. For all \( a \in K \setminus L(K, \text{Aut}(G)) \) and \( \nu \in \text{Aut}(G) \) we get \( \nu(a) = a[a, \nu] \in aS(K, \text{Aut}(G)) \). It follows that \( \text{orb}(a) \subseteq aS(K, \text{Aut}(G)) \) and hence
\[
|\text{orb}(a)| \leq |S(K, \text{Aut}(G))|
\]
for all \( a \in K \setminus L(K, \text{Aut}(G)) \). By (1.3), we have
\[
\Pr(K, \text{Aut}(G)) = \frac{1}{|K|} \left( \sum_{a \in L(K, \text{Aut}(G))} \frac{1}{|\text{orb}(a)|} + \sum_{a \in K \setminus L(K, \text{Aut}(G))} \frac{1}{|\text{orb}(a)|} \right)
\]
\[
\geq \frac{1}{|K||L(K, \text{Aut}(G))|} + \frac{1}{|K|} \sum_{a \in K \setminus L(K, \text{Aut}(G))} \frac{1}{|S(K, \text{Aut}(G))|}.
\]
Hence, the result follows.

The following corollary is a generalization of [1, Equation (3)].

Corollary 3.3. If \( K \) is a subgroup of \( G \), then
\[
\Pr(K, \text{Aut}(G)) \geq \frac{1}{|[K, \text{Aut}(G)]|} \left( 1 + \frac{|[K, \text{Aut}(G)]| - 1}{|K : L(K, \text{Aut}(G))|} \right).
\]

Proof. The result follows from Theorem 3.2 and (2.4) noting that
\[
|[K, \text{Aut}(G)]| \geq |S(K, \text{Aut}(G))|.
\]

It is clear from the above proof that Theorem 3.2 gives better lower bound than Corollary 3.3.

Also
\[
\frac{1}{|[K, \text{Aut}(G)]|} \left( 1 + \frac{|[K, \text{Aut}(G)]| - 1}{|K : L(K, \text{Aut}(G))|} \right) \geq \frac{|L(K, \text{Aut}(G))|}{|K|} + \frac{|L(K, \text{Aut}(G))|}{|K||\text{Aut}(G)|}.
\]

Hence, the lower bound given by Corollary 3.3 is better than that in [6, Theorem 2.3 (i)].

The following result is a generalization of [1, Proposition 3] which gives several equivalent conditions for equality in Corollary 3.3.

Proposition 3.4. If \( K \) is a subgroup of \( G \) then the following statements are equivalent.

(a) \( \Pr(K, \text{Aut}(G)) = \frac{1}{|[K, \text{Aut}(G)]|} \left( 1 + \frac{|[K, \text{Aut}(G)]| - 1}{|K : L(K, \text{Aut}(G))|} \right) \).

(b) \( |\text{orb}(a)| = |[K, \text{Aut}(G)]| \) for all \( a \in K \setminus L(K, \text{Aut}(G)) \).

(c) \( \text{orb}(a) = a[K, \text{Aut}(G)] \) for all \( a \in K \setminus L(K, \text{Aut}(G)) \), and so \([K, \text{Aut}(G)] \subseteq L(K, \text{Aut}(G))\).
(d) $C_{\text{Aut}(G)}(a) \triangleleft \text{Aut}(G)$ and $\frac{\text{Aut}(G)}{C_{\text{Aut}(G)}(a)} \cong [K, \text{Aut}(G)]$ for all $a \in K \setminus L(K, \text{Aut}(G))$.

(e) $[K, \text{Aut}(G)] = \{a^{-1}\nu(a) : \nu \in \text{Aut}(G)\}$ for all $a \in K \setminus L(K, \text{Aut}(G))$.

Proof. First note that for all $a \in K$

(3.2) $\text{orb}(a) \subseteq a[K, \text{Aut}(G)]$.

Suppose that (a) holds. Then, by (1.3), we have

$$\sum_{a \in K \setminus L(K, \text{Aut}(G))} \left(\frac{1}{|\text{orb}(a)|} - \frac{1}{|[K, \text{Aut}(G)]|}\right) = 0.$$  

Now using (3.2), we get (b). Also, if (b) holds then from (1.3), we have (a). Thus (a) and (b) are equivalent.

Suppose that (b) holds. Then for all $a \in K \setminus L(K, \text{Aut}(G))$ we have $|\text{orb}(a)| = |a[K, \text{Aut}(G)]|$. Hence, using (3.2) we get (c). If $[K, \text{Aut}(G)] \not\subseteq L(K, \text{Aut}(G))$ then there exist $z \in [K, \text{Aut}(G)] \setminus L(K, \text{Aut}(G))$. Therefore $\text{orb}(z) = z[K, \text{Aut}(G)] = [K, \text{Aut}(G)]$, a contradiction. Hence $[K, \text{Aut}(G)] \subseteq L(K, \text{Aut}(G))$. It can be seen that the mapping $f : \text{Aut}(G) \to [K, \text{Aut}(G)]$ given by $\nu \mapsto a^{-1}\nu(a)$, where $a$ is a fixed element of $K \setminus L(K, \text{Aut}(G))$, is a surjective homomorphism with kernel $C_{\text{Aut}(G)}(a)$. Therefore (d) follows.

Since $|\text{Aut}(G)|/|C_{\text{Aut}(G)}(a)| = |\text{orb}(a)|$ for all $a \in K \setminus L(K, \text{Aut}(G))$ we have (b).

Thus (b), (c), and (d) are equivalent.

Also $\text{orb}(a) = a[K, \text{Aut}(G)]$ if and only if $a^{-1}\text{orb}(a) = [K, \text{Aut}(G)]$ for all $a \in K \setminus L(K, \text{Aut}(G))$, which gives the equivalence of (c) and (e). This completes the proof.

Let $M_K = \max\{|\text{orb}(a)| : a \in K \setminus L(K, \text{Aut}(G))\}$. The following theorem gives a lower bound for $\Pr(K, \text{Aut}(G))$ involving $M_K$.

Theorem 3.5. If $K$ is a subgroup of $G$ then

$$\Pr(K, \text{Aut}(G)) \geq \frac{1}{M_K} \left(1 + \frac{M_K - 1}{|K : L(K, \text{Aut}(G))|}\right)$$

with equality if and only if $M_K = |\text{orb}(a)|$ for all $a \in K \setminus L(K, \text{Aut}(G))$. 


Proof. Since $|\text{orb}(a)| \leq M_K$ for all $a \in K \setminus L(K, Aut(G))$, we have

$$\sum_{a \in K \setminus L(K, Aut(G))} \frac{1}{|\text{orb}(a)|} \geq \frac{|K| - |L(K, Aut(G))|}{M_K}.$$ 

Hence, the result follows from (2.1).

For any $a \in K \setminus L(K, Aut(G))$ we have $\text{orb}(a) \subseteq aS(K, Aut(G))$ where $aS(K, Aut(G)) = \{ak : k \in S(K, Aut(G))\}$. Therefore $|S(K, Aut(G))| \geq M_K$ and hence, by (2.4), we have

$$\frac{1}{M_K} \left(1 + \frac{M_K - 1}{|K : L(K, Aut(G))|}\right) \geq \frac{1}{|S(K, Aut(G))|} \left(1 + \frac{|S(K, Aut(G))| - 1}{|K : L(K, Aut(G))|}\right).$$

This shows that Theorem 3.5 gives better lower bound than Theorem 3.2.

4. Autoisoclinism between pairs of groups

Hall [4], in the year 1940, introduced isoclinism between two groups. After many years, autoisoclinism between two groups was introduced by Moghad-dam et al. [7] in 2013. Let $G_1$ and $G_2$ be two groups. Suppose there exist isomorphisms $\phi : \frac{G_1}{L(G_1)} \rightarrow \frac{G_2}{L(G_2)}$, $\gamma : Aut(G_1) \rightarrow Aut(G_2)$ and $\beta : [G_1, Aut(G_1)] \rightarrow [G_2, Aut(G_2)]$ such that the diagram

$\begin{array}{cccc}
\frac{G_1}{L(G_1)} \times Aut(G_1) & \phi \times \gamma & \frac{G_2}{L(G_2)} \times Aut(G_2) \\
\uparrow & \downarrow \beta & \uparrow \\
[G_1, Aut(G_1)] & [G_1, Aut(G_1)] & [G_2, Aut(G_2)]
\end{array}$

commutes, where the maps $a_{(G_i, Aut(G_i))} : \frac{G_i}{L(G_i)} \times Aut(G_i) \rightarrow [G_i, Aut(G_i)]$ for $i = 1, 2$ are given by

$$a_{(G_i, Aut(G_i))}(x_iL(G_i), \nu_i) = [x_i, \nu_i].$$

Then the groups $G_1$ and $G_2$ are called autoisoclinic and the triple $(\phi, \gamma, \beta)$ is an autoisoclinism between them. A generalization of this notion of autoisoclinism between two groups is given below.

Definition 4.1. Let $K_1$ and $K_2$ be two subgroups of the groups $G_1$ and $G_2$ respectively. A pair of groups $(K_1, G_1)$ is said to be autoisoclinic to another pair of groups $(K_2, G_2)$ if there exist isomorphisms $\phi : \frac{K_1}{L(K_1, Aut(G_1))} \rightarrow \frac{K_2}{L(K_2, Aut(G_2))}$, $\gamma : Aut(K_1) \rightarrow Aut(K_2)$ and $\beta : [K_1, Aut(K_1)] \rightarrow [K_2, Aut(K_2)]$ such that the diagram

$\begin{array}{cccc}
\frac{K_1}{L(K_1)} \times Aut(K_1) & \phi \times \gamma & \frac{K_2}{L(K_2)} \times Aut(K_2) \\
\uparrow & \downarrow \beta & \uparrow \\
[K_1, Aut(K_1)] & [K_1, Aut(K_1)] & [K_2, Aut(K_2)]
\end{array}$

commutes, where the maps $a_{(K_i, Aut(K_i))} : \frac{K_i}{L(K_i)} \times Aut(K_i) \rightarrow [K_i, Aut(K_i)]$ for $i = 1, 2$ are given by

$$a_{(K_i, Aut(K_i))}(x_iL(K_i), \nu_i) = [x_i, \nu_i].$$
\[ \frac{K_2}{L(K_2, \text{Aut}(G_2))}, \gamma : \text{Aut}(G_1) \to \text{Aut}(G_2) \text{ and } \beta : [K_1, \text{Aut}(G_1)] \to [K_2, \text{Aut}(G_2)] \]
such that the diagram

\[
\begin{array}{ccc}
\frac{K_1}{L(K_1, \text{Aut}(G_1))} \times \text{Aut}(G_1) & \xrightarrow{\phi \times \gamma} & \frac{K_2}{L(K_2, \text{Aut}(G_2))} \times \text{Aut}(G_2) \\
[K_1, \text{Aut}(G_1)] & \xrightarrow{\beta} & [K_2, \text{Aut}(G_2)]
\end{array}
\]

commutes, where the maps \( a_{(K_i, \text{Aut}(G_i))} : \frac{K_i}{L(K_i, \text{Aut}(G_i))} \times \text{Aut}(G_i) \to [K_i, \text{Aut}(G_i)] \)
for \( i = 1, 2 \) are given by

\[ a_{(K_i, \text{Aut}(G_i))}(x_iL(K_i, \text{Aut}(G_i)), \nu_i) = [x_i, \nu_i]. \]

Such a triple \((\phi, \gamma, \beta)\) is said to be an autoisoclinism between the pairs \((K_1, G_1)\) and \((K_2, G_2)\).

**Theorem 4.2.** Let \( G_1 \) and \( G_2 \) be two finite groups with subgroups \( K_1 \) and \( K_2 \), respectively. If the pairs \((K_1, G_1)\) and \((K_2, G_2)\) are autoisoclinic, then

\[ \text{Pr}(K_1, \text{Aut}(G_1)) = \text{Pr}(K_2, \text{Aut}(G_2)). \]

**Proof.** Consider the sets \( S = \{(x_1L(K_1, \text{Aut}(G_1)), \nu_1) \in \frac{K_1}{L(K_1, \text{Aut}(G_1))} \times \text{Aut}(G_1) : \nu_1(x_1) = x_1 \} \) and \( T = \{(x_2L(K_2, \text{Aut}(G_2)), \nu_2) \in \frac{K_2}{L(K_2, \text{Aut}(G_2))} \times \text{Aut}(G_2) : \nu_2(x_2) = x_2 \} \). Since \((K_1, G_1)\) is autoisoclinic to \((K_2, G_2)\) we have \(|S| = |T|\). Again, it is clear that

\[ (4.1) \left| \{(x_1, \nu_1) \in K_1 \times \text{Aut}(G_1) : \nu_1(x_1) = x_1 \} \right| = |L(K_1, \text{Aut}(G_1))||S| \]

and

\[ (4.2) \left| \{(x_2, \nu_2) \in K_2 \times \text{Aut}(G_2) : \nu_2(x_2) = x_2 \} \right| = |L(K_2, \text{Aut}(G_2))||T|. \]

Hence, the result follows from (1.1), (4.1), and (4.2).

Note that Theorem 4.2 is a generalization of [10, Lemma 2.5]. We conclude the paper by noting that the bounds obtained in Section 2 and Section 3 for \text{Pr}(K, \text{Aut}(G))\ are also applicable for \text{Pr}(K_1, \text{Aut}(G_1))\ if \((K_1, G_1)\) is autoisoclinic to \((K, G)\).
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References


