On additive maps of MA-semirings with involution

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Abstract:

We extend the concept of \textsuperscript{*}-derivations of rings to a certain class of semirings called MA-semirings and establish some results on commutativity forced by the \textsuperscript{*}-derivations satisfying different criteria. We specially focus on the results on certain conditions under which additive mappings become Jordan \textsuperscript{*}-derivations. -code, which, when self-dual, produces an unimodular lattice by Construction A.

Keywords: MA-semirings; \textsuperscript{*}-semirings; \textsuperscript{*}-derivations; Jordan \textsuperscript{*}-derivations.


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1. Introduction and Preliminaries

The concept of involution is studied by many algebraists for algebras, groups, rings and other structures [5, 7, 11, 12, 13, 14, 16]. Another aspect which carries much importance in ring theory is derivation. M. Bresar and J. Vukman [6] studied the concept of *-derivation and Jordan *-derivation for rings. We can roughly say that a *-derivation is a derivation with involution. In the present paper, we canonically extend the concept of *-derivation for a class of semirings called MA-semirings introduced by Javed et al [8]. For more on MA-semirings one can see [1, 2, 3, 4, 9, 15]. We generalize some results for *-derivations of MA-semirings established in [10] for rings.

Now we include some definitions and preliminaries necessary for completion. An additive inverse semiring \( S \) with absorbing zero ‘0’ is called an MA-Semiring if \( r + r' \in Z, \forall r \in S \), where \( Z \) is the center of \( S \) and \( r' \) is the pseudo inverse of \( r \). Obviously every ring is an MA-semiring but the following example shows that converse may not be true.

**Example 1.1.** [8] The set \( S = \{0, 1, 2, 3, 4, \ldots\} \) with addition \( \oplus \) and multiplication \( \odot \) respectively defined by \( a \oplus b = \sup\{a, b\} \) and \( a \odot b = \inf\{a, b\} \) is an MA-semiring. In fact \( S \) is a commutative prime MA-semiring.

Such examples motivate us to generalize the results of ring theory for MA-semirings. Throughout the paper, by a *-semiring \( S \), we mean a *-MA-semiring unless stated otherwise. \( S \) is prime if \( aRb = 0 \) implies that \( a = 0 \) or \( b = 0 \). \( S \) is semiprime if \( aRa = 0 \) implies that \( a = 0 \). An additive mapping \( \ast : S \rightarrow S \) is involution if \( \forall u, v \in S, (uv)^\ast = v^\ast u^\ast \).

By a *-semiring we simply mean a semiring \( S \) with involution \( \ast \). Following example describes a *-MA-semiring.

**Example 1.2.** If \((R, +, \cdot)\) is an MA-Semiring, then the set \( R \) with addition ‘\( + \)’ and multiplication ‘\( \cdot \)’ defined as \( a \cdot b = b \cdot a \) forms an MA-semiring called the opposite MA-Semiring of \( R \). We usually notate it as \( R^o \).

Let \((R, +, \cdot)\) be an MA-semiring and \( R^o \) its opposite MA-semiring. Consider \( S = R \times R^o \) with \( (a, b) \oplus (c, d) = (a+c, b+d) \) and \( (a, b) \odot (c, d) = (a.c, b.d) = (ac, db) \). Then \((S, \oplus, \odot)\) forms an MA-semiring. Define \( \ast : S \rightarrow S \) by \( (x, y)^\ast = (y, x) \). Therefore \( \ast \) defines an involution on \( S \) and \((S, \oplus, \odot)\) forms a \( \ast \)-MA-semiring or MA-semiring with involution.

\( S \) is 2-torsion free if for \( u \in S, 2u = 0 \) implies \( u = 0 \) and 3-torsion if \( 3u = 0 \) implies \( u = 0 \). An additive mapping \( d : S \rightarrow S \) is said to be a
derivation if \(d(uv) = d(u)v + ud(v)\). A \(*\)-derivation is an additive mapping \(d : S \rightarrow S\) such that \(d(uv) = d(u)v^* + ud(v)\). By Jordan \(*\)-derivation, we mean an additive mapping \(d : S \rightarrow S\) satisfying \(d(u^2) = d(u)u^* + ud(u)\).

An additive mapping \(F : S \rightarrow S\) is generalized derivation associated with a derivation \(d\) if \(F(xy) = F(x)y + xd(y)\). We define Commutator as \([u, v] = uv + v^* u\). By Jordan product we mean \(u \circ v = uv + yu\) for all \(u, v \in S\).

Following identities will be used frequently: \([u, uv] = u[u, v], [uv, w] = u[v, w] + [u, w]v, [u, yw] = [u, v]w + v[u, w], [u, v] + [v, u] = v(u + u^*) = u(v + v^*), (uv)^* = u^* v = uv^*, [u, v]^* = [u, v]^* = [u^*, v], u \circ (v + w) = u \circ v + u \circ w\) (see [8],[15]).

2. Main Results

**Theorem 2.1.** Let \(S\) be a semiprime \(*\)-semiring. If \(T\) is an additive mapping satisfying

\[
\tag{2.1} T(uv) + T(u^*)v^* = 0, \forall u, v \in S
\]

Then \([T(S), S] = 0\) and hence \(T(S) \subseteq Z(S)\).

**Proof.** In (2.1) writing \(uw\) for \(u\), we get \(\forall u, v \in S\)

\[
\tag{2.2} T(uwv) + T(u^*)v^* = 0
\]

and \(wv\) for \(v\) in (2.2), we get \(T(uwv) + T(u^*wv^*) = 0\), which implies

\[
\tag{2.3} T(uwv) + T(u^*)v^*w^* = 0
\]

From (2.3), put \(T(uwv) = T(uw)v^*, \) we get

\[
\tag{2.4} T(uw)v^* + T(u^*)v^*w^* = 0
\]

From (2.1) using \(T(uv) = T(u)v^*\) into (4), we obtain \(T(u)w^*v^* + T(u^*)v^*w^* = 0\), which implies \(T(u)w^*v^* + T(u^*)v^*w^* = 0\) and therefore \(T(u)(w^*v^* + v^*w^*) = 0\), which further gives

\[
\tag{2.5} T(u)[w^*, v^*] = 0
\]

In (2.5) replacing \(w\) by \(w^*\) and \(v\) by \(v^*\), we obtain

\[
\tag{2.6} T(u)[w, v] = 0
\]
In (2.6), replacing \( w \) by \( w^T(u) \), we obtain \( T(u)[wT(u), v] = 0 \), which implies \( T(u)[T(u), v] + T(u)[w, v]T(u) = 0 \). Using (2.6) again, we obtain

\[
T(u)w[T(u), v] = 0
\]

Replacing \( w \) by \( vw \) in (2.7), we obtain

\[
T(u)vw[T(u), v] = 0
\]

Multiplying (2.7) by \( v^0 \) from the left, we obtain

\[
v^0T(u)w[T(u), v] = 0
\]

Adding (2.8) and (2.9), we obtain \( T(S, v)wT(S, v) = 0 \) which implies \( T(S, v)S[T(S, v), v] = 0 \). Since \( S \) is semiprime, therefore the last equation yields \( T(S, v) = 0 \) and hence \( T(S) \subseteq Z(S) \).

**Theorem 2.2.** Let \( S \) be prime *-semiring. If \( S \) admits a nontrivial *-derivation \( d \) such that \( d(uv) + d(u')d(v) = 0 \), \( \forall u, v \in S \), then \( d = 0 \).

**Proof.** By hypothesis for all \( u, v \in S \), we have

\[
d(uv) + d(u')d(v) = 0 \tag{2.10}
\]

By definition of *-derivation, from (2.10), we obtain

\[
d(u)v^* + ud(v) + d(u')d(v) = 0 \tag{2.11}
\]

In (2.11) replacing \( u \) by \( uw \), we obtain \( d(uw)v^* + uwd(v) + d(u'w)d(v) = 0 \) and again using (2.11), we obtain \( d(u)d(w)v^* + xwd(v) + d(u'w)d(v) = 0 \), which after simplification implies \( (d(u)d(w)+d(u')d(w))v^*+(u+d(u'))wd(v) = 0 \). Using (10), we obtain \( (u+d(u'))wd(v) = 0 \) and therefore \( (u+d(u'))Sd(v) = 0 \). As \( S \) is prime, either \( (u+d(u')) = 0 \) or \( d(v) = 0 \). If \( (u+d(u')) = 0 \), then \( u = d(u) \), a contradiction, which shows that \( d(v) = 0 \) and therefore \( d = 0 \).

**Theorem 2.3.** Let \( S \) be prime *-semiring. If \( S \) admits a *-derivation \( d \) such that \( d \neq I^* \) and \( d(uv) + d(v)d(u') = 0 \), \( \forall u, v \in S \), then \( d = 0 \) (where \( I^*(u) = u^* \)).
Proof. By the hypothesis for all \( u, v \in S \)

\[
(2.12) \quad d(u, v) + d(v)d(u') = 0
\]

In (2.12) writing \( uv \) for \( v \), we obtain \( d(uuv) + d(uv)d(u') = 0 \) which further gives on simplification \( d(u)\nu^*(u^* + d(u')) + u(d(uv) + d(v)d(u')) = 0 \). Using (2.12) again, we obtain \( d(u)\nu^*(u^* + d(u')) = 0 \), which implies \( d(u)S(u^* + d(u')) = 0 \). By the primeness of \( S \), we have either \( u^* + d(u') = 0 \) or \( d(u) = 0 \). If \( u^* + d(u') = 0 \), then \( d(u') = u^* = I^*(u) \), which implies that \( d = I^* \), a contradiction. Therefore we obtain \( d(u) = 0 \) and \( d = 0 \) as required.

Theorem 2.4. Let \( S \) be prime \(*\)-semiring and \( a \in S \). If \( S \) admits a \(*\)-derivation \( d \) such that \([d(u), a] = 0 \) \( \forall u \in S \), then \( a \in Z(S) \) or \( d(a) = 0 \).

Proof. We have for all \( u \in S \)

\[
(2.13) \quad [d(u), a] = 0
\]

In (13) replacing \( u \) by \( uv \), we obtain \([d(uv), a] = 0 \). On simplification, we obtain \( d(u)[v^*, a] + [d(u), a]v^* + u[d(v), a] + [u, a]d(v) = 0 \). Using (13), again, we obtain

\[
(2.14) \quad d(u)[v^*, a] + [u, a]d(v) = 0
\]

Replacing \( u \) by \( a \) in (2.14), we obtain \( d(a)[v^*, a] + [a, a]d(v) = 0 \) and therefore

\[
(2.15) \quad d(a)[v^*, a] + a(d(v)a + d(v)a') = 0
\]

From (2.13), replacing \( u \) by \( v \), we obtain \( d(v)a = ad(v) \), and hence using it in (2.15), we have \( d(a)[v^*, a] + a[d(v), a] = 0 \). Using (2.13) again, we have

\[
(2.16) \quad d(a)[v^*, a] = 0
\]

Replacing \( v \) by \( v^* \), we obtain

\[
(2.17) \quad d(a)[v, a] = 0
\]

In (2.17), replacing \( v \) by \( vu \) and using it again, we obtain \( d(a)S[u, a] = 0 \). By the primeness of \( S \), we have \( d(a) = 0 \) or \( [u, a] = 0 \) and therefore \( d(a) = 0 \) or \( a \in Z(S) \).

Theorem 2.5. Let \( S \) be semiprime \(*\)-semiring. If \( S \) admits a \(*\)-derivation \( d \) such that \( d[u, v] = 0 \), then \( d = 0 \) or \( S \) is commutative.
Proof. We have for all $u, v \in S$

$$d[u, v] = 0 \tag{2.18}$$

Replacing $u$ by $uv$ in (2.18), we obtain $d[uv, v] = 0$ and therefore $d[u, v]v^* + [u, v]d(v) = 0$. Using (2.18) again, we obtain

$$[u, v]d(v) = 0 \tag{2.19}$$

Replacing $u$ by $su$ in (2.19), we obtain $[su, v]d(v) = 0$ which implies $s[u, v]d(v) + [s, v]ud(v) = 0$. Using (2.19) again, we obtain $[s, v]ud(v) = 0$ and therefore

$$[s, v]Rd(v) = 0 \tag{2.20}$$

By primeness of $S$, (2.20) yields either $[s, v] = 0$ or $d(v) = 0$. Now take $K = \{v \in S : d(v) = 0\}$ and $L = \{v \in S : [s, v] = 0, \forall s \in S\}$. Clearly $S = K \cup L$. We claim that either $S = K$ or $S = L$. For this we can show that either $L \subseteq K$ or $K \subseteq L$. Suppose that $u \in K \setminus L$ and $v \in L \setminus K$. Clearly $u + v \in K + L \subseteq S = K \cup L$. Therefore $u + v \in K$ or $u + v \in L$. Firstly, If $u + v \in K$, then $d(u + v) = 0$ which implies $d(u) + d(v) = 0$ and therefore $d(v) = 0$ which means $v \in K$, a contradiction. Secondly, if $u + v \in L$ $[u + v, r] = [u, r] + [v, r] = [u, r] = 0, \forall r \in S$, which implies $u \in L$, a contradiction. Therefore, we have either $L \subseteq K$ or $K \subseteq L$ and hence either $S = K$ or $S = L$. This proves that that either $d = 0$ or $S$ is commutative. □

**Theorem 2.6.** Let $S$ be prime *-semiring. If $S$ admits a *-derivation $d$ such that $d(u \circ v) = 0$, $\forall u, v \in S$, then $d = 0$ or $S$ is commutative.

Proof. For any $u, v \in S$, We have

$$d(u \circ v) = 0 \tag{2.21}$$

In (2.21) replacing $u$ by $uv$, we obtain $d((uv) \circ v) = 0$. But $d((uv) \circ v) = d(u \circ v)v$. Therefore $d(u \circ v)v = 0$ and hence $d(u \circ v)v^* + (u \circ v)d(v) = 0$. Using (2.21) again, we obtain

$$d(u \circ v)d(v) = 0 \tag{2.22}$$

In (2.22) replacing $u$ by $sv$, we obtain $((sv) \circ v)d(v) = 0$, which implies
Let $K$. Proof. We have for any such that $S = \{v \in S : s \circ v = \emptyset, \forall s \in S\}$. Clearly $S = K \cup L$. Our claim is that either $S = K$ or $S = L$. For this we show that either $K \subseteq L$ or $L \subseteq K$. Suppose that $u \in K \setminus L$ and $v \in L \setminus K$. Clearly $u + v \in K + L \subseteq S = K \cup L$, which implies $u + v \in K$ or $u + v \in L$. Firstly, If $u + v \in K$, then $d(u + v) = d(u) + d(v) = d(v) = 0$ which means $v \in K$, a contradiction. Secondly, if $u + v \in L$, then $r \circ (u + v) = r \circ u + r \circ v = r \circ u = 0, \forall r \in S$, which means $u \in L$, a contradiction. Therefore we obtain either $L \subseteq K$ or $K \subseteq L$, which implies that either $S = K$ or $S = L$. If $S = K$, then $d = 0$. On the other hand, if $S = L$, then for any $s, v \in S$

\begin{equation}
(2.23) \quad (s \circ v)Sd(v) = 0
\end{equation}

Since $S$ is prime, therefore (2.23) yields either $(s \circ v) = 0$ or $d(v) = 0$. Let $K = \{v \in S : d(v) = 0\}$ and $L = \{v \in S : s \circ v = \emptyset, \forall s \in S\}$. Clearly $S = K \cup L$. Then $s \circ v = 0$. Using (26), we obtain $[v, s]w = 0$. This proves that $S$ is commutative. \hfill \square

**Theorem 2.7.** Let $S$ be prime *-semiring. If $S$ admits a *-derivation $d$ such that $d(u) \circ v = 0, \forall u, v \in S$, then $d = 0$ or $S$ is commutative.

**Proof.** We have for any $u, v \in S$

\begin{equation}
(2.24) \quad s \circ v = 0
\end{equation}

In (2.24) replacing $s$ by $sw$, we obtain $(sw) \circ v = 0$, which implies $svw + vsw = 0$. Since $s = s + s' + s$ and $s + s' \in Z(S)$ therefore last equation becomes $svw + v(s + s' + s)w = 0$ which gives on simplification that $s(w \circ v) + [v, s]w = 0$. Using (2.24) again, we obtain $[v, s]w = 0$. Replacing $w$ by $uw$, we obtain $[v, s]Su = 0$. By the primeness of $S$, since $S \neq 0$, we obtain $[v, s] = 0$. This proves that $S$ is commutative.

\begin{equation}
(2.25) \quad d(u) \circ v = 0
\end{equation}

In (2.25) replacing $u$ by $uw$, we obtain $(d(u)w^* + ud(w)) \circ v = 0$. Since $v + v' \in Z, v + v' + v = v$ and $v + v + v' = v'$, after simplification we obtain

\begin{equation}
(2.26) \quad (d(u) \circ v)w^* + d(u)[w^*, v] + u(v \circ d(w)) + [v, u]d(w) = 0
\end{equation}

Using (26), we obtain

\begin{equation}
(2.27) \quad d(u)[w^*, v] + [v, u]d(w) = 0
\end{equation}

Replacing $u$ by $v$, (2.27), we obtain $d(v)[w^*, v] + [v, v]d(w) = 0$. Using the definition of $S$ and simplifying we obtain


\[(2.28)\quad d(v)[w^*, v] + v(d(w) + v^*d(w)) = 0\]

From (2.25), we have \(d(w)v = v'd(w)\). Hence (2.28) becomes \(d(v)[w^*, v] + v(d(w) \circ v) = 0\). Using (2.25) again, we obtain

\[(2.29)\quad d(v)[w^*, v] = 0\]

In (2.29) replacing \(w\) by \(w^*\), we obtain

\[(2.30)\quad d(v)[w, v] = 0\]

Replacing \(w\) by \(uw\) in (2.30), we obtain \(d(v)[uw, v] = 0\) which further implies \(d(v)u[w, v] + [d(u), v]w = 0\). Using (2.30) again, we obtain

\[(2.31)\quad d(v)S[w, v] = 0\]

Since \(S\) is prime, therefore from (2.31), we have \(d(v) = 0\) or \([w, v] = 0\). The remaining part is same as that of Theorem 2.5.

**Theorem 2.8.** Let \(S\) be a 2-torsion free semiprime *-semiring. Suppose that \(au^*b^* + bua = 0, \forall u \in S\), for some \(a, b \in S\). Then \(ab = 0 = ba\). Moreover if \(S\) is prime, then either \(a = 0\) or \(b = 0\).

**Proof.** By the hypothesis

\[(2.32)\quad au^*b^* + bua = 0\]

In (2.32) replacing \(u\) by \(vbu\), we obtain \(a(vbu)^*b^* + bua = 0\)

\[(2.33)\quad au^*b^*v^*b^* + bvba = 0\]

From (2.32), using \(au^*b^* = bua^*\) into (2.33), we obtain \(buav^*b^* + bvba = 0\) which further implies

\[(2.34)\quad buva + bvba = 0\]

In particular for \(v = u\), we obtain \(2buba = 0, \forall u \in S\) and 2-torsion freeness of \(S\) further yields

\[(2.35)\quad buba = 0\]

Again from (2.32), using \(au^*b^* = bua^*\) into (2.35), we obtain
(2.36) \[ bua^*b^* = 0 \]

In (2.35), replacing \( v \) by \( uav \), we obtain \( bub(uav)a + b(uav)bua = 0 \). Using (2.32), we obtain \( (bua^*b^*)'va + buabua = 0 \). Using (2.36) again, we obtain \( buaSbua = 0 \) and therefore by the semiprimeness, we obtain

(2.37) \[ bua = 0 \]

This implies \( abuab = 0 \). By the semiprimeness of \( S \), we have \( ab = 0 \).

Again from (2.37), we have \( bauba = 0 \), which implies \( ba = 0 \). Hence we conclude that \( ba = 0 = ba \). Moreover if \( S \) is prime then (2.37) yields either \( a = 0 \) or \( b = 0 \). \( \square \)

**Theorem 2.9.** Let \( S \) be a 2-torsion free semiprime \(*\)-semiring and \( F : S \to S \) be an additive mapping satisfying

(2.38) \[ F(u^*v^*) + F(u^*v^*) + u^*F(v^*) + uvf(u) = 0, \forall u, v \in S \]

associated with the Jordan \(*\)-derivation \( f \). Then \( F \) is a Jordan \(*\)-derivation.

**Proof.** Replacing \( u \) by \( u + w \) by in (2.38), we obtain

\[ F((u+w)v^*(u+w)) + F(u+w)v^*(u+w)^* + (u+w)f(v)(u+w)^* + (u+w)vf(u+w) = 0 \]

which further implies

\[ F(uv^*u) + F(uv^*u) + F(uv^*u) + F(uv^*u) + F(v^*u^*) + F(u^*v^*) + f(v^*) + wf(u^*) + u^*F(v^*) + uvf(u) + uvf(u) + uvf(u) + uvf(u) = 0. \]

Using (2.38) again we obtain

\[ F(uv^*u) + F(uv^*u) + F(uv^*u) = 0 \]

In (2.39), replacing \( w \) by \( u^2 \), we obtain

\[ F(uw^*u) + F(uw^*u) + F(u^2v^*u^2) + F(u^2v^*u^2) + u^2f(v)u^* + uf(v)u^* + u^2vf(u) + uvf(u^2) = 0 \]

Replacing \( v \) by \( uv + vu \) in (2.38), we obtain

\[ F(u(uv + vu)^* u) + F(u(uv + vu)^* u^* + uf(uv + vu)u^* + u((uv + vu))f(u) = 0, \]
which further implies
\[ F(u^2 v' u + uv' u^2) + F(u)u^* v^* u^* + F(u)v^* u^2 \]

\[ (2.41) \]

From (2.40), we have
\[ F(u^2 v^* u^2) = F(u^2 v^* u^2) \]

\[ (2.42) \]

Using (2.42) into (2.41), we obtain
\[ F(u)u^* v^* u^* + F(u^2 v^* u^2) = 0 \]

(2.42) into (2.41), we obtain
\[ (F(u^2) + (F(u))^2) + u v f(u) = A(u) \]

(2.44) in (2.43), we obtain
\[ (F(u^2) + (F(u))^2) u v f(u) = 0 \]

(2.45) in (2.44), we obtain
\[ A(u)v^* u^* = 0 \]

Replacing \( v \) by \( v^* \) in (2.44), we obtain
\[ A(u)v^* u^* = 0 \]

(2.46) in (2.45), we obtain
\[ A(v)u^* u^* = 0 \]

Replacing \( v \) by \( u^* v A(u) \), we obtain \( A(u)u^* RA(u)u^* = 0 \) and by the semiprimeness, we get
\[ (2.47) \]

In (2.47) replacing \( u \) by \( u + v \), we obtain \( A(u + v)(u + v)^* = 0 \), which further implies \( A(u) + B(u, v) + A(v) = 0 \), where \( B(u, v) = F(u^2 v^* u^2) + (F(u^2) + u v f(u) + (F(u))^2) \). Hence we have
\[ A(u)u^* + B(u, v)u^* + A(v)u^* + A(u)v^* + B(u, v)v^* + A(v)v^* = 0 \] Using (2.47) again, we obtain
\[ (2.48) \]

\[ B(u, v)u^* + A(v)u^* + A(u)v^* + B(u, v)v^* = 0 \]
In (2.48) replacing $u$ by $u'$, we obtain $B(u', v)u^* + A(v)u^* + A(u')v^* + B(u', v)v^* = 0$, which further implies $B(u, v)u^* + (A(v)u^*)' + A(u)v^* + (B(u, v)v^*)' = 0$ and hence

$$B(u, v)u^* + A(u)v^* = A(v)u^* + B(u, v)v^*$$

Using (2.49) into (2.48), we obtain $2(B(u, v)u^* + A(u)v^*) = 0$ and by 2-torsion freeness of $S$, we obtain

$$B(u, v)u^* + A(u)v^* = 0$$

Multiplying (2.50) by $A(u)$ from the right, we obtain $B(u, v)u^*A(u) + A(u)v^*A(u) = 0$. Using (2.46), we obtain $A(u)v^*A(u) = 0$. Replacing $v$ by $v^*$, we obtain $A(u)RA(u) = 0$. By the semiprimeness of $S$, we obtain $A(u) = 0$. Therefore $F(u^2) + (F(u))'u^* + u'F(u) = 0$, which further implies $F(u^2) = F(u)u^* + uF(u)$ and this completes the proof. □

**Theorem 2.10.** Let $S$ be a 2-torsion and 3-torsion free semiprime *-semiring and $D : S \longrightarrow S$ be an additive mapping satisfying

$$D(uv'u) + D(u)v^*u^* + uD(v)u^* + uvD(u) = 0, \forall u, v \in S$$

Then $D$ is a Jordan *-derivation.

**Proof.** In (2.50), replacing $u$ by $u^2$, we obtain

$$D(u^2v'u^2) + D(u^2)v^*u^2 + u^2D(v)u^2 + u^2vD(u^2) = 0$$

In (2.51) replacing $v$ by $uvu$, we obtain $D(u^2v'u^2) + D(u)u^*v^*u^2 + uD(uvu)u^* + u(uvu)D(u) = 0$. Using (2.51) into the last equation, we obtain $D(u^2v'u^2) + D(u)u^*v^*u^2 + u(D(u)v^*u^* + uD(v)u^* + uvD(u))u^* + u(uvu)D(u) = 0$. Therefore

$$D(u^2v'u^2) + D(u)u^*v^*u^2 + uD(u)v^*u^2 + u^2D(v)u^2 + u^2vD(u)u^* + u^2vuD(u) = 0$$

(2.53)

Since $v + v' \in Z$, $v + v' + v = v$, $v' + v + v' = v'$, therefore from (2.52), we have

$$D(u^2v'u^2) + u^2D(v)u^2 = D(u^2)v^*u^2 + u^2vD(u^2)$$

(2.54)
Using (2.54) into (2.53), we obtain \[ D(u)v u^* u^* u^2 + u D(u)v u^* u^2 + u^2 v D(u)u^* + u^2 v u D(u) + D(u^2) v u^* u^2 + u^2 v D(u^2) = 0. \] This further implies \[ u^2 v' (D(u^2) + D(u)u^* + u' D(u))v' u^2 = 0 \] and therefore \[ u^2 v (D(u^2) + D(u)u^* + u' D(u)) + (D(u^2) + D(u)u^* + u' D(u))v' u^2 = 0. \] Setting \[ A(u) = D(u^2) D(u)u^* + u' D(u) \] into the last equation, we obtain

\[(2.55) \quad u^2 v A(u) + A(u) v u^2 = 0 \]

In view of Theorem 2.8, we can write

\[(2.56) \quad A(u) u^2 = 0 \]

\[(2.57) \quad u^2 A(u) = 0 \]

linearizing (2.56), we obtain

\[(2.58) \quad A(u + v) (u + v)^2 = 0 \]

We can easily see that \[ A(u + v) = A(u) + B(u, v) + A(v), \] where \[ B(u, v) = D(uv + vu) + (D(u))(v^* + (D(v))) u^* + u D(v) + v' D(u). \] Hence (2.59) becomes

\[(2.59) \quad B(u, v)u^2 + A(v) u^2 + A(u) v^2 + B(u, v) v^2 + A(v) v^2 + A(u) (uv + vu) + B(u, v)(uv + vu) + A(v)(uv + vu) = 0. \]

Using (2.56) again in the last equation, we obtain

\[(2.60) \quad u^2 + B(u, v)(uv + vu) + A(v)(uv + vu) = 0 \]

We can easily observe that \[ A(u') = A(u) \] and \[ B(u', v) = (B(u, v)). \] Replacing \( u \) by \( u' \) in (2.59), we obtain

\[(2.60) \quad u^2 + B(u, v)(uv + vu) + A(v)(uv + vu) = 0 \]

From (2.60), we have

\[(2.61) \quad (B(u, v)) u^2 + (B(u, v)) v^2 + A(u)(uv + vu) + A(v)(uv + vu) = 0 \]

Using (2.61) into (2.59), we obtain 2((B(u, v)) u^2 + (B(u, v)) v^2 + A(u)(uv + vu) + A(v)(uv + vu)) = 0. Since \( S \) is 2-torsion free, therefore
(2.62) \[ B(u,v)u^2 + B(u,v)v^2 + A(u)(uv + vu) + A(v)(uv + vu) = 0 \]

We can easily see that \( A(2u) = 4A(u) \) and \( B(2u, v) = 2B(u, v) \). Replacing \( u \) by \( 2u \) in (2.62), we obtain \( 8B(u,v)u^2 + 2B(u,v)v^2 + 8A(u)(uv + vu) + 2A(v)(uv + vu) = 0 \), which can also be written as \( 2(4B(u,v)u^2 + B(u,v)v^2 + 4A(u)(uv + vu) + A(v)(uv + vu)) = 0 \). By the 2-torsion freeness of \( S \), we obtain

(2.63) \[ 4B(u,v)u^2 + B(u,v)v^2 + 4A(u)(uv + vu) + A(v)(uv + vu) = 0 \]

Since \( v + v' \in Z \), \( v + v' + v = v \), \( v' + v + v' = v' \), therefore from (2.62), we have

(2.64) \[ B(u,v)v^2 + A(v)(uv + vu)) = (B(u,v))'(u^2 + (A(u))'(uv + vu) \]

Using (2.64) into (2.63), we obtain \( 4B(u,v)u^2 + (B(u,v))'(u^2 + 4A(u)(uv + vu) + (A(u))'(uv + vu) = 0 \). Since \( u + u' + u = u, u' + u + u' = u' \), therefore \( 3B(u,v)u^2 + 3A(u)(uv + vu) = 0 \) and hence by 3-torsion freeness of \( S \), we have

(2.65) \[ B(u,v)u^2 + A(u)(uv + vu) = 0 \]

Multiplying (2.65) by \( A(u)v \) from the right and using (2.57), we obtain

(2.66) \[ A(u)vuvA(u)v + A(vv)uA(u)v = 0 \]

In (2.66), replacing \( v \) by \( vu \), we obtain \( A(u)vuvA(u)v + A(vv)uA(u)v = 0 \). Using (2.56), we obtain \( A(u)vuvA(u)v = 0 \), which further implies \( uA(u)vvuA(u)v = 0 \). By the semiprimeness of \( S \), we obtain

(2.67) \[ A(u)v = 0 \]

Hence (2.65) becomes

(2.68) \[ B(u,v)u^2 + A(u)vu = 0 \]

Multiplying (2.68) by \( A(u) \) from the right and using (2.57), we obtain \( A(u)vvuA(u) = 0 \), which implies \( uA(u)vvuA(u) = 0 \) and hence \( A(u)vuA(u) = 0 \) and by the semiprimeness, we have
(2.69) \[ uA(u) = 0 \]

From (2.68), we have \( (B(u, v)u + A(u)v)v = 0 \), which implies \( (B(u, v)u + A(u)v)v = 0 \) and therefore

(2.70) \[ B(u, v)u + A(u)v = 0 \]

Multiplying (2.70) by \( A(u) \) from the right, we obtain \( B(u, v)uA(u) + A(u)vA(u) = 0 \) and using (2.70) again, we obtain \( A(u)vA(u) = 0 \). Since \( S \) is semiprime, therefore \( A(u) = 0 \). This means \( D(u^2) + D(u)u^2 + u^*D(u) = 0 \) and hence \( D(u^2) = D(u)u^* + uD(u) \), which shows that \( D \) is Jordan \( * \)-derivation. \( \square \)

Concluding Remarks

This article presents some criteria for \( * \)-derivations which induce commutativity in additive inverse semirings with involution. Secondly we present some additive mappings satisfying certain conditions under which they become Jordan \( * \)-derivations. Therefore ideas presented in this article are useful. We propose some open problems as follows:

1. Let \( S \) be a semiprime \( * \)-semiring and \( d \) a nonzero \( * \)-derivation of \( S \) satisfying \( d(u) \circ v = 0, \forall u, v \in S \). Is \( S \) commutative?

2. Let \( S \) be a semiprime \( * \)-semiring and \( d \) a nonzero \( * \)-derivation of \( S \) satisfying \( d(u \circ v) = 0, \forall u, v \in S \). Is \( S \) commutative?

3. Let \( S \) be a prime \( * \)-semiring, \( d \) a nonzero \( * \)-derivation of \( S \) and \( F \) an additive mapping defined by \( F(xy) = F(x)y + xd(y) \). If \( F \) satisfies \( F(u \circ v) = 0, \forall u, v \in S \). Is \( S \) commutative?

References


