Irregularity indices for line graph of Dutch windmill graph

Mohanad A. Mohammed\textsuperscript{1} \textsuperscript{orcid.org/0000-0002-7507-1212}
Suad Younus A. AL-Mayyahi\textsuperscript{2}
Abaid ur Rehman Virk\textsuperscript{3} \textsuperscript{orcid.org/0000-0001-7312-9225}
Hafiz Mutee ur Rehman\textsuperscript{4} \textsuperscript{orcid.org/0000-0002-7496-5804}

\textsuperscript{1}Open Educational College, Dept. of Mathematics, Al-Qudsia Centre, Ministry of Education, Amarah, Iraq
\texttt{mohanadalim@gmail.com}
\textsuperscript{2}University of Wasit, Dept. of Mathematics, Faculty of Education for Pure Sciences, Wasit, Kut, Iraq
\texttt{suadhis3@gmail.com}
\textsuperscript{3}University of Management and Technology Lahore, Dept. of Mathematics, Pakistan.
\texttt{abaid.math@gmail.com}
\textsuperscript{4}University of Education Lahore, Dept. of Mathematics, Pakistan.
\texttt{rehman.mutee@yahoo.com}
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Abstract:

Among topological descriptors topological indices are significant and they have a conspicuous role in chemistry. Dutch Windmill graph $D_n$ can be obtain by taking $x$ copies of cycle $C_3$ with a vertex in common. In this paper, we will compute some irregularity indices that are useful in quantitative structure activity relationship for Line Graph of Dutch Windmill graph.

Keywords: Dutch Windmill graph; Irregularity indices.

MSC (2020): 05C10, 05C12, 05C15, 05C22, 05C31.

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1. Introduction

There are lot of curious real life issues that can be deciphered in the language of graph theory, where they are often found to have attractive solutions. Let $G = (V, E)$ be a simple connected graph, $V$ is the set of vertices and $E$ represents the number of edges present in graph. Degree of vertex means how many edges are attached with that vertex and is denoted by $d_v$ where $v \in V(G)$ and $e$ represents an edge $e = uv \in E(G)$. Topological indices (TIs) help us to describe the structure of graph [2,3,8,9,10,11,12,14,15,25]. First ever TI was presented by Winer in 1947 [31], when he was trying to find out the boiling point of alkanes.

$$W(G) = \sum_{(u, v) \subseteq V(G)} d_G(u, v)$$

In 1975, Gutman gave a remarkable identity [32] about Zagreb indices. Hence, these two indices are among the oldest degree-based descriptors and their properties are extensively investigated. The mathematical formulae of these indices are:

$$M_1(G) = \sum_{uv \in E(G)} (d_u + d_v),$$

$$M_2(G) = \sum_{uv \in E(G)} (d_u \times d_v).$$

Historically, Zagreb indices are the very first degree based TIs, but these indices were used for completely different purpose, therefore the first genuine degree based TI is Randić index which was given in 1975 by Milan Randić [27] as:

$$R(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_u.d_v}}.$$ 

An unexpected mathematical quality of Randić index is discovered recently, that tells us about the relation of this topological invariant with normalized Laplacian Matric [1,5,24]. The general Randić index [19] is defined as:

$$GRI(G) = \sum_{uv \in E(G)} (d_u.d_v)^{\alpha}.$$ 

where $\alpha$ is an arbitrary real number. In Graph theory, the Line graph of a graph $G$ is represented by $L(G)$ that represents the adjacencies between the edges of $G$. The most important theorems about Line graphs is presented
Irregularity indices for line graph of Dutch windmill graph

by Hassler Whitney [30] in (1932), he proved that with one exceptional case
the structure of graph $G$ can be recovered completely from its Line graph.

A streamlined method of expressing the irregularity of graph is the
The TI is known as Irregularity index, [28] if TI of graph is greater equal
to zero and TI of graph is equal to zero if and only if graph is regular. The
Irregularity indices are given below.

- $\text{VAR}(G) = \sum_{uv \in V} (d_u - \frac{2m}{n})^2 = \frac{M_1(G)}{n} - (\frac{2m}{n})^2$
- $\text{AL}(G) = \sum_{uv \in \mathcal{E}(G)} |d_u - d_v|$
- $\text{IR}_1(G) = \sum_{u \in V} (d_u)^3 - \frac{2m}{n} \sum_{u \in V} (d_u)^2 = F(G) - \frac{2m}{n}\overline{M_1}(G)$
- $\text{IR}_2(G) = \sqrt{\frac{\sum_{uv \in \mathcal{E}(G)} d_ud_v}{m}} - \frac{2m}{n} = \sqrt{\frac{M_2(G)}{m}} - \frac{2m}{n}$
- $\text{IRF}(G) = \sum_{uv \in \mathcal{E}(G)} (d_u - d_v)^2 = F(G) - 2M_2(G)$
- $\text{IRFW}(G) = \frac{\text{IRF}(G)}{M_2(G)}$
- $\text{IRA}(G) = \sum_{uv \in \mathcal{E}(G)} (d_u^{-1/2} - d_v^{-1/2})^2 = n - 2R(G)$
- $\text{IRB}(G) = \sum_{uv \in \mathcal{E}(G)} (d_u^{1/2} - d_v^{1/2})^2 = M_1(G) - 2RR(G)$
- $\text{IRC}(G) = \frac{\sum_{uv \in \mathcal{E}(G)} \sqrt{d_ud_v}}{m} - \frac{2m}{n} = \frac{R(G)}{m} - \frac{2m}{n}$
- $\text{IRDIF}(G) = \sum_{uv \in \mathcal{E}(G)} \sqrt{d_u - d_v} = \sum_{i<j} m_{i,j}(\frac{j}{i} - \frac{i}{j})$
- $\text{IRL}(G) = \sum_{uv \in \mathcal{E}(G)} |\ln d_u - \ln d_v| = \sum_{i<j} m_{i,j} \ln(\frac{j}{i})$
- $\text{IRLU}(G) = \sum_{uv \in \mathcal{E}(G)} \frac{|d_u - d_v|}{\min(d_u, d_v)} = \sum_{i<j} m_{i,j} \ln(\frac{j}{i})$
- $\text{IRLF}(G) = \sum_{uv \in \mathcal{E}(G)} \frac{|d_u - d_v|}{\sqrt{d_u d_v}} = \sum_{i<j} m_{i,j} (\frac{j-i}{\sqrt{ij}})$
\[ \text{IRLA}(G) = 2 \sum_{uv \in E(G)} \frac{|d_u - d_v|}{(d_u + d_v)} = 2 \sum_{i < j} m_{i,j}(\frac{i - j}{i + j}) \]

\[ \text{IRD1}(G) = \sum_{uv \in E(G)} \ln 1 + |d_u - d_v| = \sum_{i < j} m_{i,j} \ln(i + j - 1) \]

\[ \text{IRGA}(G) = \sum_{uv \in E(G)} \ln \left( \frac{d_u + d_v}{2\sqrt{d_u d_v}} \right) \sum_{i < j} m_{i,j} \left( \frac{i + j}{2\sqrt{ij}} \right) \]

For more about TIs one can study [7,13,16,17,18,20,21,22,23,26,29].

2. Irregularity indices for Line Graph of Dutch Windmill Graph

A graph \( D_y^x \) with \( x \geq 1 \) and \( y \geq 3 \) is known as Dutch Windmill Graph [4]. \( D_y^x \) can be obtain by taking \( x \) copies of cycle \( C_y \) with a vertex in common. Figure 1(a) shows the Dutch Windmill Graph \( D_y^x \) with \( x = 4 \) and \( y = 4 \) and Figure 1(b) shows the line graph of Dutch Windmill Graph \( D_y^x \) with \( x = 4 \) and \( y = 4 \). We can observe that the order of \( L(D_y^x) \) is \( xy \) and size of \( L(D_y^x) \) is \( 2x^2 + xy - 2x \). We, now portioned the edge set according to their degrees. There are three types of edges present in line graph of Dutch Windmill Graph \( E_{(2,2)}, E_{(2,2x)}\) and \( E_{(2x,2x)}\). The frequencies of these edges are given in Table 1.

![Figure 2.1: (a) \( D_4^4 \), (b) \( L(D_4^4) \)]](image)
<table>
<thead>
<tr>
<th>((d_u, d_v))</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2, 2)</td>
<td>(x(y - 3))</td>
</tr>
<tr>
<td>(2, 2x)</td>
<td>(2x)</td>
</tr>
<tr>
<td>(2x, 2x)</td>
<td>((2x - 1)x)</td>
</tr>
</tbody>
</table>

Table 2.1: Partition of \(E(D_y^x)\)

Let \(G\) be the Line Graph of Dutch Windmill Graph \(D_y^x\) for \(x \geq 1\) and \(y \geq 3\), then we have

1. \(VAR(G) = \frac{4}{y^2}(2x^2y - x^2 - 2xy + 2x - 1).\)

2. \(AL(G) = 4x(1 - x).\)

3. \(IR1(G) = \frac{8x}{y}(2x^3y - 2x^3 - 2x^2y + 2x^2 - xy + 2x + y - 2).\)

4. \(IR2(G) = 2\sqrt{\frac{2x^3 - x^2 + 2x + y - 3}{x^2 + xy - x}} - \frac{x^2 + xy - x}{xy} \cdot \frac{2x}{x^2 + xy - x}.\)

5. \(IRF(G) = 8x(x^2 - 2x + 1).\)

**Proof.**

\[
VAR(G) = \sum_{u \in V(G)} \left( d_u - \frac{2m}{n} \right)^2 \\
= \frac{M_1(G)}{n} - \left( \frac{2m}{n} \right)^2 \\
= \frac{4x(2x^2 + y - 2)}{xy} - \left( \frac{2x(y - 3)}{xy} \right)^2 \\
= \frac{4}{(y)^2} \left( 2x^2y - x^2 - 2xy + 2x - 1. \right)
\]

\[
AL(G) = \sum_{u \in V(G)} |d_u - d_v| \\
= |2 - 2|(x(y - 3)) + |2 - 3x|(2x) + |2x - 2x|(x(2x - 1)) \\
= 4x(1 - x).
\]
\[ IR1(G) = \sum_{u \in V} d_u^3 - \frac{2m}{n} \sum_{u \in V} d_u^2 \\
= F(G) - \left( \frac{2m}{n} \right) M_1(G) \\
= 8x(2x + 2\sqrt{x} + xy - 4) - \left( \frac{2(x^2 + xy - x)}{xy} \right)(4x(2x^2 + y - 2)) \\
= \frac{8x}{y}(2x^3y - 2x^3 - 2x^2y + 2x^2 - xy + 2x + y - 2). \]

\[ IR2(G) = \sqrt{\frac{\sum_{u \in V} d_u d_v}{m}} - \frac{2m}{n} \\
= \sqrt{\frac{M_2(G)}{m}} - \frac{2m}{n} \\
= \sqrt{\frac{4x(2x^3 - x^2 + 2x + y - 3)}{(x^2 + xy - x)}} - \frac{2(x^2 + xy - x)}{xy} \\
= 2 \left( \sqrt{\frac{2x^3 - x^2 + 2x + y - 3}{x^2 + xy - x}} - \frac{x^2 + xy - x}{xy} \right). \]

\[ IRF(G) = \sum_{u \in V} (d_u - d_v)^2 \\
= (12 - 2)^2(xy - 3x) + (2 - 2x)^2(xy) + (2x - 2x)^2(x(2x - 1)) \\
= 8x(x^2 - 2x + 1). \]

\[ \square \]

Let \( G \) be the Line Graph of Dutch Windmill Graph \( D_y^x \) for \( x \geq 1 \) and \( y \geq 3 \), then we have

1. \( IRW(G) = \frac{2x^2 - 2x + 1}{2x^2 - x^2 + 2x + y - 3} \)
2. \( IRA(G) = 4 - 2x - 2\sqrt{x} \).
3. \( IRB(G) = 4x(\sqrt{x} - 1)^2 \).
4. \( IRC(G) = \frac{2}{xy(x^2 - x + xy)}(2x^5/2y + 2x^4y - x^4 - x^3y + x^2y^2 + 2x^3 - 2x^2y - 3x^2y - x^2 + 2x^2y - xy^2) \).
5. \( IRDIF(G) = 2 - 2x^2 \).
Proof.

\[ IRFW(G) = \frac{IRF(G)}{M_2(G)} = \frac{2(x^2 - 2x + 1)}{2x^3 - x^2 + 2x + y - 3}. \]

\[ IRA(G) = \sum_{uv \in E(G)} (d_u^{-1/2} - d_v^{-1/2})^2 \]
\[ = n - 2R(G) \]
\[ = (xy) - 2\left( \frac{1}{2}(2x + 2\sqrt{x} + xy - 4) \right) \]
\[ = 4 - 2x - 2\sqrt{x}. \]

\[ IRB(G) = \sum_{uv \in E(G)} (d_u^{1/2} - d_v^{1/2})^2 \]
\[ = M_1(G) - 2RR(G) \]
\[ = (\sqrt{2} - \sqrt{2})^2(x(y - 3)) + (\sqrt{2} - \sqrt{2})^2(xy) + (\sqrt{2x} + \sqrt{2x})^2(x(2x - 1)) \]
\[ = 4x(\sqrt{x} - 1)^2. \]

\[ IRC(G) = \sum_{uv \in E(G)} \sqrt{d_u d_v} - \frac{2m}{n} \]
\[ = \frac{RR(G) - 2m}{n} \]
\[ = \frac{2(x^2 - x + xy)}{nx^2 - x + xy} \]
\[ = \frac{2x^2y - 3x^2y - x^2 + 2x^2y - xy^2).} \]

\[ IRDIF(G) = \sum_{uv \in E(G)} \left| \frac{d_u}{d_v} - \frac{d_v}{d_u} \right| \]
\[ = \left| \frac{2}{2} - \frac{2}{2}(xy - 3x) + \frac{2}{2x} - \frac{2}{2} \right|(xy) + \frac{2}{2x} - \frac{2}{2x}(2x^2 - x) \]
\[ = 2 - 2x^2. \]
Let $G$ be the Line Graph of Dutch Windmill Graph $D^x_y$ for $x \geq 1$ and $y \geq 3$, then we have

1. $IRL(G) = -2\ln(x)x$.
2. $IRLF(G) = (1 - x)\sqrt{xy}$.
3. $IRLA(G) = \frac{4x(1-x)}{4+x}$.
4. $IRD1(G) = 4x(1 - x)$.
5. $IRGA(G) = 2\ln\left(\frac{2+2x}{4\sqrt{x}}\right)$.

Proof.

$$IRL(G) = \sum_{uv \in E(G)} |\ln d_u - \ln d_v|$$

$$= |\ln 2 - \ln 2|(xy - 3) + |\ln 2 - \ln(2x)|(xy) + |\ln(2x) - \ln(2x)|(2x^2 - x)$$

$$= -2\ln(x)x.$$

$$IRLF(G) = \sum_{uv \in E(G)} \frac{|d_u - d_v|}{\sqrt{d_u.d_v}}$$

$$= \left(\frac{|2 - 2|}{\sqrt{2}}\right)(xy - 3) + \left(\frac{|2 - 2x|}{\sqrt{4x}}\right)(xy) + \left(\frac{|2x - 2x|}{\sqrt{4x^2}}\right)(2x^2 - x)$$

$$= (1 - x)\sqrt{xy}.$$

$$IRLA(G) = \sum_{uv \in E(G)} \frac{2|d_u - d_v|}{(d_u + d_v)}$$

$$= 2 \left(\frac{|2 - 2|}{2 + 2}\right)(xy - 3) + 2 \left(\frac{|2 - 2x|}{2 + 2x}\right)(xy) + 2 \left(\frac{|2x - 2x|}{2x + 2x}\right)(2x^2 - x)$$

$$= \frac{4x(1-x)}{1+x}.$$
\[ IRD1(G) = \sum_{uv \in E(G)} \ln\{1 + |d_u - d_v|\} \]
\[ = \ln\{1 + |2 - 2|\}(xy - 3x) + \ln\{1 + |2 - 2x|\}(xy) \]
\[ + \ln\{1 + |2x - 2x|\}(2x^2 - x) \]
\[ = 4x(1 - x). \]

\[ IRGA(G) = \sum_{uv \in E(G)} \ln\left(\frac{d_u + d_v}{2\sqrt{d_u d_v}}\right) \]
\[ = \ln\left(\frac{2 + 2}{2\sqrt{2 \times 2}}\right)(xy - 3x) + \ln\left(\frac{2 + 2x}{2\sqrt{2x \times 2x}}\right)(xy) \]
\[ + \ln\left(\frac{2x + 2x}{2\sqrt{2x \times 2x}}\right)(2x^2 - x) \]
\[ = 2\ln\left(\frac{2 + 2x}{4\sqrt{x}}\right). \]
3. Graphical Representation
Irregularity indices for line graph of Dutch windmill graph
Conclusions

In this paper, we calculate sixteen irregularity indices for Line Graph of Dutch Windmill Graph $L(D_y^x)$. TIs are numeric quantities that help us to study different parameters of underlines structure. TIs are used for the development of quantitative structure-activity relationships (QSARs).

Acknowledgements

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References


