Proyecciones (Antofagasta. On line) | vol. 39, n. 5 (2020) | pp. 1201-1220.

REVISTAS CIENTÍFICAS de la Universidad Católica del Norte. revistas.ucn.cl

doi 10.22199/issn.0717-6279-2020-05-0074

PROYECCIONES Journal of Mathematics

ISSN 0717-6279 (On line)

On functions of $(\phi, 2, \alpha)$-bounded variation

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Received: July 2019 | Accepted: August 2019

Abstract:

We introduce the $(\phi, 2, \alpha)$-bounded variation spaces, which are a common generalization between Riesz's spaces, p-variation and $(\phi, 2)$-bounded variation spaces. We also study its structure as Banach spaces, as well as some embedding results.

Keywords: Riesz p-variation; $(\phi, 2)$-bounded variation; Bounded variation.

MSC (2020): 26A45, 26B30, 26A16, 26A24.

Cite this article as (IEEE citation style):

R. E. Castillo, H. C. Chaparro, and E. Trousselot, "On functions of $(\phi, 2, \alpha)$-bounded variation", *Proyecciones (Antofagasta, On line)*, vol. 39, no. 5, pp. 1201-1220, Oct. 2020, doi: 10.22199/issn.0717-6279-2020-05-0074.

1. Introduction

The concept of variation was introduced by C. Jordan [5] around 1880, while he was studying convergence of Fourier series. Since then, this con- cept has been generalized in many ways by many mathematicians. For one-variable functions, there are a lot of results about Banach space struc- ture and embeddings of the so called Bounded Variation Spaces, while for multivariable functions, there is no standard definition of variation. The interested reader is invited to check [1] for some of the history and basic results about this topic.

Some of those generalizations were inspired by problems in areas such as mathematical physics [8], calculus of variations, convergence of Fourier series and geometric measure theory, among others. Moreover, those spaces find applications in information theory, data compression and signal processing [4]. So, it is natural and interesting to continue with the study of spaces with some type of variation.

Usually, when one refers to generalizations of the classical Lebesgue L_p spaces, and also p-variation spaces, it is typical to work with a function ϕ which has the most relevant features of $f(t) = t^p$ for $t \geq 0$ and $p \geq 1$. In this setting, a ϕ-function is a function $\phi : [0, \infty) \to [0, \infty)$ which is continuous, strictly increasing, $\phi(t) = 0$ if and only if $t = 0$ and such that $\lim_{t\to\infty} \phi(t) = \infty$. In addition, for a convex ϕ-function, if $\lim_{t\to\infty} \phi(t)/t = \infty$, then we say that ϕ satisfies the ∞_1-condition.

In his 1991 paper [6], N. Merentes generalized the notion of $(p, 2)$-bounded variation with $p \geq 1$. He introduced the concept of $(\phi, 2)$-bounded variation in the sense of Riesz, in the following way: Let ϕ be a ϕ-function. A function $f : [a, b] \to \mathbf{R}$ has $(\phi, 2)$-bounded variation on $[a, b]$ if the number

$$\begin{aligned} & V^R_{(\phi,2)}(f) \\ &= V^R_{(\phi,2)}(f; [a, b]) \\ &= \sup_\Pi \sum_{j=1}^{n-1} \phi\left(\left|\frac{f(x_{j+1})-f(x_j)}{(x_{j+1}-x_j)(x_{j+1}-x_{j-1})} - \frac{f(x_j)-f(x_{j-1})}{(x_j-x_{j-1})(x_{j+1}-x_{j-1})}\right|\right) \\ &\quad \times |x_{j+1} - x_{j-1}| \end{aligned}$$

is finite, where the supremum is taken over all partitions $\Pi : a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b]$. The class of functions of $(\phi, 2)$-bounded variation, which is denoted by $V^R_{(\phi,2)}([a, b])$ is not necessarily a linear space. N. Merentes also showed that if ϕ is convex and satisfies the ∞_1-condition, then all functions of $(\phi, 2)$-bounded variation have bounded second variation (see [6, Lemma 2.1]).

On the other hand, the first and third named authors, together with H.

Rafeiro, introduced the $(2, \alpha)$-variation in their paper [3]. They also proved some results on equivalent definitions, embeddings and Banach structure for spaces with this type of variation.

In this paper, we generalize all this, by introducing a strictly increasing and continuous function α to obtain the $(\phi,\ 2,\ \alpha)$-bounded variation in the sense of Riesz. We consider in the whole paper partitions by blocks, since it can be generalized to the (ϕ, k, α) bounded variation (see [7]). The set of these functions is denoted by $V^R_{(\phi,2,\alpha)}([a, b])$.

This paper is organized as follows. In Section 2 we give some basic definitions and results that will be useful in the coming sections. In Section 3, we prove an embedding result and some properties of the operator $V^R_{(\phi,2,\alpha)}(\cdot)$. Since $V^R_{(\phi,2,\alpha)}([a, b])$, in general, is not a vector space, those results allow us to define the linear space $\mathrm{RBV}_{(\phi,2,\alpha)}$, the space generated by $V^R_{(\phi,2,\alpha)}([a, b])$. In Section 4 we prove that the space $\mathrm{RBV}_{(\phi,2,\alpha)}$ is a normed space, and we use this to prove some embedding results in Section 5. Finally, in Section 6, we show that $\mathrm{RBV}_{(\phi,2,\alpha)}$ is actually a Banach space.

2. Preliminaries

We gather some useful results which will be necessary throughout the paper.

Lemma 2.1. *Let $(\mathbf{R}, V, +)$ be a linear space. Let $A \subset V$ be a symmetric and convex subset. Let $[A]$ be the linear space generated by A. Then*

$$[A] = \{v \in V : \exists\ \lambda > 0 \text{ such that } \lambda v \in A\}.$$

Lemma 2.2. *Let ϕ be a convex ϕ-function defined on $[0, \infty)$. Then the function $\psi : (0, \infty) \to \mathbf{R}$ defined by*

$$\psi(x) = \frac{\phi(x)}{x}$$

is non-decreasing.

Definition 2.3. *Let f be a real function defined on $[a, b]$ and Π a block partition of $[a, b]$, that is*

$\Pi : a = x_{1,1} < x_{1,2} \le x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \le x_{2,3} < x_{2,4} = x_{3,1} < \cdots < x_{n-1,4} = x_{n,1} < x_{n,2} \le x_{n,3} < x_{n,4} = b.$

Let

$$\sigma^{(2,\alpha)}(f, \Pi) = \sum_{j=1}^{n-2} |f_\alpha[x_{j,1}, x_{j,2}] - f_\alpha[x_{j,3}, x_{j,4}]|\,,$$

where

$$f_\alpha[p,q] = \frac{f(q)-f(p)}{\alpha(q)-\alpha(p)},$$

and

$$V^{(2,\alpha)}(f,[a,b]) = V^{(2,\alpha)}(f) = \sup_{\Pi} \sigma^{(2,\alpha)}(f,\Pi),$$

where the supremum is taken over all partitions of $[a,b]$.

The number $V^{(2,\alpha)}(f)$ *is called* $(2,\alpha)$*-variation in the sense of de la Valle Poussin of* f *on the interval* $[a,b]$*. If* $V^{(2,\alpha)}(f) < \infty$*, we say that* f *is of* $(2,\alpha)$*-bounded variation in the sense of de la Valle Poussin. The set of all such functions is denoted by* $RV^{(2,\alpha)}([a,b])$.

As customary, $B([a,b])$ denotes the set of all bounded functions on $[a,b]$.

Definition 2.4. *Let* f *and* α *be real-valued functions defined on the same open interval* I *and let* $x_0 \in I$*. If the limit*

$$\lim_{x\to x_0} \frac{f(x)-f(x_0)}{\alpha(x)-\alpha(x_0)}$$

exists, we say that f *is* α*-differentiable at* x_0*, and we denote its value by* $f'_\alpha(x_0)$.

Definition 2.5. *Let* $f \in RV^{(2,\alpha)}([a,b])$*, we define* $\|\cdot\|_{\mathrm{RV}^{(2,\alpha)}([a,b])}$ *by*

$$\|f\|_{\mathrm{RV}^{(2,\alpha)}([a,b])} = |f(a)| + |f'_\alpha(a)| + V^{(2,\alpha)}(f).$$

It was shown in [3, Section 6] that $\left(\mathrm{RV}^{(2,\alpha)}([a,b]), \|\cdot\|_{\mathrm{RV}^{(2,\alpha)}([a,b])}\right)$ is a normed space.

3. $\mathbf{RV}_{(\phi,2,\alpha)}([a,b])$ as a linear space

We introduce now the $(\phi,2,\alpha)$-bounded variation of a function. As said in the introduction, this is a common generalization between $(\phi,2)$-bounded variation and $(2,\alpha)$-bounded variation. In what follows, unless stated otherwise, α stands for any strictly increasing continuous function defined on $[a,b]$.

Definition 3.1. *Let* ϕ *be a* ϕ*-function and* f *a real function defined on* $[a,b]$*. Let* Π *be a block partition of the interval* $[a,b]$,

$\Pi : a = x_{1,1} < x_{1,2} \leq x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \leq x_{2,3} < x_{2,4} = x_{3,1} < \cdots < x_{n-1,4} = x_{n,1} < x_{n,2} \leq x_{n,3} < x_{n,4} = b.$

Let

$$\begin{aligned}
&\sigma^R_{(\phi,2,\alpha)}(f,\Pi)\\
&= \sum_{j=1}^n \phi\left(\left|\frac{f(x_{j,4})-f(x_{j,3})}{(\alpha(x_{j,4})-\alpha(x_{j,3}))(\alpha(x_{j,4})-\alpha(x_{j,1}))}\right.\right.\\
&\quad\left.\left.-\frac{f(x_{j,2})-f(x_{j,1})}{(\alpha(x_{j,2})-\alpha(x_{j,1}))(\alpha(x_{j,4})-\alpha(x_{j,1}))}\right|\right) \times |\alpha(x_{j,4})-\alpha(x_{j,1})|\\
&= \sum_{j=1}^n \phi\left(\frac{\left|\frac{f(x_{j,4})-f(x_{j,3})}{\alpha(x_{j,4})-\alpha(x_{j,3})}-\frac{f(x_{j,2})-f(x_{j,1})}{\alpha(x_{j,2})-\alpha(x_{j,1})}\right|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) \times |\alpha(x_{j,4})-\alpha(x_{j,1})|\\
&= \sum_{j=1}^n \phi\left(\frac{|f_\alpha[x_{j,4};x_{j,3}]-f_\alpha[x_{j,2};x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) \times |\alpha(x_{j,4})-\alpha(x_{j,1})|.
\end{aligned}$$

And

$$V^R_{(\phi,2,\alpha)}(f,[a,b]) = V^R_{(\phi,2,\alpha)}(f) = \sup_\Pi \sigma^R_{(\phi,2,\alpha)}(f,\Pi),$$

where the supremum is taken over all possible block partition of $[a,b]$. $V^R_{(\phi,2,\alpha)}(f)$ is called the $(\phi,2,\alpha)$-variation of f in the sense of Riesz on the interval $[a,b]$. If $V^R_{(\phi,2,\alpha)}(f) < \infty$, the function f is said to be of $(\phi,2,\alpha)$-bounded variation in the sense of Riesz. The set of all this functions is denoted by $BV^R_{(\phi,2,\alpha)}([a,b])$.

Remark 3.2. *It is observed that, if $\phi(t) = t^p$, with $t \geq 0$ and $p \geq 1$ and $\alpha(t) = t$, then we obtain $V^R_{(\phi,2,\alpha)}([a,b]) = V^R_{(\phi,2)}([a,b])$, that is, the $(\phi,2,\alpha)$-bounded variation generalizes the $(\phi,2)$-bounded variation in the sense of Riesz.*

In the next theorem, we prove that every function with $(\phi,2,\alpha)$-bounded variation is also of $(2,\alpha)$-bounded variation.

Theorem 3.3. *Let ϕ be a convex ϕ-function. If $f \in V^R_{(\phi,2,\alpha)}([a,b])$, then $f \in RV^{(2,\alpha)}([a,b])$. Moreover,*

$$V^{(2,\alpha)}(f) \leq \frac{1}{\phi(1)} V^R_{(\phi,2,\alpha)}(f) + \alpha(b) - \alpha(a).$$

Proof. Let

$\Pi : a = x_{1,1} < x_{1,2} \leq x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \leq x_{2,3} < x_{2,4} = x_{3,1} < \cdots < x_{n-1,4} = x_{n,1} < x_{n,2} \leq x_{n,3} < x_{n,4} = b.$

be a partition of $[a,b]$. Let

$$E = \{j \in \{1,\ldots,n\} : |f_\alpha[x_{j,4},x_{j,3}] - f_\alpha[x_{j,2},x_{j,1}]| \geq |\alpha(x_{j,4}) - \alpha(x_{j,1})|\}.$$

If $j \notin E$, then

$$|f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]| < |\alpha(x_{j,4}) - \alpha(x_{j,1})|.$$

Now, if $j \in E$, then

$$\frac{|f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]|}{|\alpha(x_{j,4}) - \alpha(x_{j,1})|} \geq 1,$$

and by Lemma 2.2, we obtain

$$\frac{\phi(1)}{1} \leq \frac{\phi\left(\frac{|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right)}{\frac{|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}},$$

therefore

$$\begin{aligned} |f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]| &\leq \tfrac{1}{\phi(1)}\phi\left(\tfrac{|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) \\ &\quad \times |\alpha(x_{j,4}) - \alpha(x_{j,1})|. \end{aligned}$$

Thus, let us consider

$$\begin{aligned} &\sum_{j=1}^n \left|\tfrac{f(x_{j,4})-f(x_{j,3})}{\alpha(x_{j,4})-\alpha(x_{j,3})} - \tfrac{f(x_{j,2})-f(x_{j,1})}{\alpha(x_{j,2})-\alpha(x_{j,1})}\right| \\ &= \sum_{j=1}^n |f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]| \\ &= \sum_{j\in E} |f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]| + \sum_{j\notin E} |f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]| \\ &\leq \tfrac{1}{\phi(1)} \sum_{j\in E} \phi\left(\tfrac{|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) \times |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\ &\quad + \sum_{j\notin E} |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\ &< \tfrac{1}{\phi(1)} \sum_{j\in E} \phi\left(\tfrac{|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) \times |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\ &\quad + \sum_{j=1}^n |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\ &\leq \tfrac{1}{\phi(1)} V^R_{(\phi,2,\alpha)}(f) + \alpha(b) - \alpha(a). \end{aligned}$$

In this way we obtain

$$\sigma^{(2,\alpha)}(f, \Pi) \leq \frac{1}{\phi(1)} V^R_{(\phi,2,\alpha)}(f) + \alpha(b) - \alpha(a) < \infty,$$

for all partitions Π of $[a, b]$. Finally,

$$V^{(2,\alpha)}(f) \leq \frac{1}{\phi(1)} V^R_{(\phi,2,\alpha)}(f) + \alpha(b) - \alpha(a) < \infty.$$

□

As a conclusion from the above result and [3, Corollary 20], we get that

$$\mathrm{BV}^R_{(\phi,2,\alpha)}([a, b]) \subset B([a, b]),$$

that is, every $(\phi, 2, \alpha)$-bounded variation function is a bounded function.

For the following theorem, we consider $V^R_{(\phi,2,\alpha)}(f)$ as a functional defined on the $\mathrm{BV}^R_{(\phi,2,\alpha)}([a, b])$ space. Compare with [2, Theorem 4.1].

Theorem 3.4. *Let*

$$\begin{array}{ll} V^R_{(\phi,2,\alpha)} : BV^R_{(\phi,2,\alpha)}([a,b]) & \to \mathbf{R} \\ f & \mapsto V^R_{(\phi,2,\alpha)}(f) \end{array}$$

and let ϕ be a ϕ-function. Then,

1. *$V^R_{(\phi,2,\alpha)}(-f) = V^R_{(\phi,2,\alpha)}(f)$, $f \in BV^R_{(\phi,2,\alpha)}([a,b])$.*
2. *If ϕ is convex, then $V^R_{(\phi,2,\alpha)}$ is convex.*
3. *If $V^R_{(\phi,2,\alpha)}(f) = 0$, then $f(x) = \lambda\alpha(x) + \mu$, $\lambda, \mu \in \mathbf{R}$.*
4. *If ϕ is convex and $0 \leq \lambda \leq 1$, then $V^R_{(\phi,2,\alpha)}(\lambda f) \leq \lambda V^R_{(\phi,2,\alpha)}(f)$, $f \in BV^R_{(\phi,2,\alpha)}([a,b])$.*

Proof.

1. It is immediate from the definition.
2. Let $f, g \in \mathrm{BV}^R_{(\phi,2,\alpha)}([a,b])$ and $\lambda, \mu \in [0,1]$ such that $\lambda + \mu = 1$. Let

   $\Pi : a = x_{1,1} < x_{1,2} \leq x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \leq x_{2,3} < x_{2,4} = x_{3,1} < \cdots < x_{n-1,4} = x_{n,1} < x_{n,2} \leq x_{n,3} < x_{n,4} = b.$

   be a partition of $[a,b]$. Let us consider

   $$\begin{aligned} &\sigma^R_{(\phi,2,\alpha)}(\lambda f + \mu g, \Pi) \\ &= \sum_{j=1}^n \phi\left(\frac{|(\lambda f+\mu g)_\alpha[x_{j,4};x_{j,3}] - (\lambda f+\mu g)_\alpha[x_{j,2};x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) \times |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\ &= \sum_{j=1}^n \phi\left(\frac{|\lambda(f_\alpha[x_{j,4};x_{j,3}] - f_\alpha[x_{j,2};x_{j,1}]) + \mu(g_\alpha[x_{j,4};x_{j,3}] - g_\alpha[x_{j,2};x_{j,1}])|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) \\ &\quad \times |\alpha(x_{j,4}) - \alpha(x_{j,1})| . \end{aligned}$$

   Now, by the convexity of ϕ, we have

   $$\begin{aligned} &\sigma^R_{(\phi,2,\alpha)}(\lambda f + \mu g, \Pi) \\ &\leq \lambda \sum_{j=1}^n \phi\left(\frac{|f_\alpha[x_{j,4};x_{j,3}] - f_\alpha[x_{j,2};x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) \times |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\ &\quad + \mu \sum_{j=1}^n \phi\left(\frac{|g_\alpha[x_{j,4};x_{j,3}] - g_\alpha[x_{j,2};x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) \times |\alpha(x_{j,4}) - \alpha(x_{j,1})| \\ &= \lambda \sigma^R_{(\phi,2,\alpha)}(f, \Pi) + \mu \sigma^R_{(\phi,2,\alpha)}(g, \Pi) \\ &\leq \lambda V^R_{(\phi,2,\alpha)}(f) + \mu V^R_{(\phi,2,\alpha)}(g) < \infty, \end{aligned}$$

   for all partitions of $[a,b]$. Thus, we have proved

   $$V^R_{(\phi,2,\alpha)}(\lambda f + \mu g) \leq \lambda V^R_{(\phi,2,\alpha)}(f) + \mu V^R_{(\phi,2,\alpha)}(g),$$

   which tells us that $V^R_{(\phi,2,\alpha)}$ is convex. Moreover, if $f, g \in \mathrm{BV}^R_{(\phi,2,\alpha)}([a,b])$, then $\lambda f + \mu g \in \mathrm{BV}^R_{(\phi,2,\alpha)}([a,b])$, with $\lambda, \mu \in [0,1]$ such that $\lambda + \mu = 1$.

3. Suppose $V^R_{(\phi,2,\alpha)}(f) = 0$ and consider any partition Π of $[a, b]$, then we have $\sigma^R_{(\phi,2,\alpha)}(f, \Pi) = 0$.

   Let Π_0 be a particular partition given by $a < t < b$, hence

$$\sigma^R_{(\phi,2,\alpha)}(f, \Pi_0) = \phi\left(\frac{\left|\frac{f(t)-f(a)}{\alpha(t)-\alpha(a)} - \frac{f(b)-f(t)}{\alpha(b)-\alpha(t)}\right|}{|\alpha(b)-\alpha(a)|}\right) \times |\alpha(b)-\alpha(a)| = 0,$$

   where

$$\frac{f(t)-f(a)}{\alpha(t)-\alpha(a)} - \frac{f(b)-f(t)}{\alpha(b)-\alpha(t)} = 0, \quad (\text{since } \phi(0) = 0)$$

   and

$$f(t) = \frac{f(t)-f(a)}{\alpha(b)-\alpha(a)}\alpha(t) + \frac{f(a)\alpha(b)-f(b)\alpha(a)}{\alpha(b)-\alpha(a)}.$$

   Taking

$$\lambda = \frac{f(b)-f(a)}{\alpha(b)-\alpha(a)} \quad \text{and} \quad \mu = \frac{f(a)\alpha(b)-f(b)\alpha(a)}{\alpha(b)-\alpha(a)}$$

   the result follows.

4. .

$$\begin{aligned} V^R_{(\phi,2,\alpha)}(\lambda f) &= V^R_{(\phi,2,\alpha)}(\lambda f + (1-\lambda)0) \\ &\leq \lambda V^R_{(\phi,2,\alpha)}(f) + (1-\lambda)V^R_{(\phi,2,\alpha)}(0) \\ &= \lambda V^R_{(\phi,2,\alpha)}(f), \quad \text{since } V^R_{(\phi,2,\alpha)}(0) = 0. \end{aligned}$$

□

Considering that, in general, $\mathrm{BV}^R_{(\phi,2,\alpha)}([a, b])$ is not a vector space, we will use Lemma 2.1 and Theorem 3.4 to obtain the $\left[\mathrm{BV}^R_{(\phi,2,\alpha)}([a, b])\right]$ space, i.e. the linear space generated by $\mathrm{BV}^R_{(\phi,2,\alpha)}([a, b])$. We state this result as follows.

Theorem 3.5. *Let ϕ be a convex ϕ-function. Then $BV^R_{(\phi,2,\alpha)}([a, b]) \subset B([a, b])$ and $BV^R_{(\phi,2,\alpha)}([a, b])$ is a symmetrical and convex set. Moreover,*

$$\left\{f : [a, b] \to \mathbf{R} : \exists\lambda > 0 \text{ such that } \lambda f \in BV^R_{(\phi,2,\alpha)}([a, b])\right\}$$

is the linear space generated by $BV^R_{(\phi,2,\alpha)}([a, b])$.

Now, by means of the previous theorem, we are ready to define the $\mathrm{RBV}_{(\phi,2,\alpha)}([a,b])$ spaces.

Definition 3.6. *Let ϕ be a convex ϕ-function. Then*

$$\begin{aligned}&\left\{f:[a,b]\to\mathbf{R}:\exists\lambda>0 \text{ such that } \lambda f\in BV^R_{(\phi,2,\alpha)}([a,b])\right\}\\ =&\left\{f:[a,b]\to\mathbf{R}:\exists\lambda>0 \text{ such that } V^R_{(\phi,2,\alpha)}(\lambda f)<+\infty\right\}\end{aligned}$$

is called the linear space of functions of $(\phi, 2, \alpha)$-bounded variation in the sense of Riesz, and it is denoted by $RBV_{(\phi,2,\alpha)}([a,b])$.

As a conclusion, we have

$$\mathrm{RBV}_{(\phi,2,\alpha)}([a,b])=\left[\mathrm{BV}^R_{(\phi,2,\alpha)}([a,b])\right]\subset B([a,b]).$$

4. $\mathbf{RBV}_{(\phi,2,\alpha)}([a,b])$ as a normed space

We will first introduce a norm which vanishes at a. We leave to the reader the proof of the following lemma. Compare with [2, Theorem 5.1].

Lemma 4.1. *Let ϕ be a convex ϕ-function. Let us consider $f\in RBV_{(\phi,2,\alpha)}([a,b])$, then we obtain the following:*

1. *If $0<k<k'$, then $V^R_{(\phi,2,\alpha)}(kf)\le V^R_{(\phi,2,\alpha)}(k'f)$.*
2. *$\lim_{\beta\to 0} V^R_{(\phi,2,\alpha)}(\beta f)=0$.*
3. *$\left\{\varepsilon>0: V^R_{(\phi,2,\alpha)}\left(\frac{f}{\varepsilon}\right)\le 1\right\}\neq\emptyset$.*

This allows us to guarantee the existence of $\inf\left\{\varepsilon>0: V^R_{(\phi,2,\alpha)}\left(\frac{f}{\varepsilon}\right)\le 1\right\}$.

Definition 4.2. *Let ϕ be a convex ϕ-function. Then*

$$RBV^0_{(\phi,2,\alpha)}([a,b])=\left\{f:[a,b]\to\mathbf{R}: f\in RBV_{(\phi,2,\alpha)}([a,b]) \text{ and } f(a)=0\right\}$$

is called the linear space of functions of $(\phi, 2, \alpha)$-bounded variation in the sense of Riesz which vanish at a.

Let us denote

$$\begin{aligned}|\cdot|^0_{(\phi,2,\alpha)}:\mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b]) &\to \mathbf{R}^+\\ f &\mapsto |f|^0_{(\phi,2,\alpha)}=\inf\left\{\varepsilon>0: V^R_{(\phi,2,\alpha)}\left(\frac{f}{\varepsilon}\right)\le 1\right\}.\end{aligned}$$

In the next result, we gather some properties of the functional $|\cdot|^0_{(\phi,2,\alpha)}$, which is well-defined by Lemma 4.1.

Lemma 4.3. *Let ϕ be a convex ϕ-function and $f \in RBV^0_{(\phi,2,\alpha)}([a,b])$. Then*

1. $|f|^0_{(\phi,2,\alpha)} \neq 0 \Rightarrow V^R_{(\phi,2,\alpha)}\left(\frac{f}{|f|^0_{(\phi,2,\alpha)}}\right) \leq 1$.
2. $|f|^0_{(\phi,2,\alpha)} < k \Rightarrow V^R_{(\phi,2,\alpha)}\left(\frac{f}{k}\right) \leq 1,\ k > 0$.
3. $0 \leq |f|^0_{(\phi,2,\alpha)} \leq 1 \Rightarrow V^R_{(\phi,2,\alpha)}(f) \leq |f|^0_{(\phi,2,\alpha)}$.

The proof of Lemma 4.3 goes line by line as the proof of Lemma 5.3 in [2].

Theorem 4.4. *Let ϕ be a convex ϕ-function. Then:*

1. $|\lambda f|^0_{(\phi,2,\alpha)} = |\lambda||f|^0_{(\phi,2,\alpha)},\ f \in RV^0_{(\phi,2,\alpha)}([a,b]),\ \lambda \in \mathbf{R}$.
2. $|f+g|^0_{(\phi,2,\alpha)} \leq |f|^0_{(\phi,2,\alpha)} + |g|^0_{(\phi,2,\alpha)},\ f,g \in RV^0_{(\phi,2,\alpha)}([a,b])$.

We omit the proof of the above theorem, since it is similar to the proof of Theorem 5.4 in [2].

Definition 4.5. *Let ϕ be a convex ϕ-function.*

$$\begin{array}{rcl} |\cdot|^R_{(\phi,2,\alpha)} : RBV^0_{(\phi,2,\alpha)}([a,b]) & \to & \mathbf{R}^+ \\ f & \mapsto & |f|^R_{(\phi,2,\alpha)} = |f'_\alpha(a)| + |f|^0_{(\phi,2,\alpha)} \\ & = & |f'_\alpha(a)| + \inf\left\{\varepsilon > 0 : V^R_{(\phi,2,\alpha)}\left(\frac{f}{\varepsilon}\right) \leq 1\right\}. \end{array}$$

Theorem 4.6. *Let ϕ be a convex ϕ-function. Then $|\cdot|^R_{(\phi,2,\alpha)}$ is a norm on $RBV^0_{(\phi,2,\alpha)}([a,b])$.*

Proof.

1. By definition, it is clear that $|f|^R_{(\phi,2,\alpha)} \geq 0$ for all $f \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$.
2. .
$$\begin{aligned} |\lambda f|^R_{(\phi,2,\alpha)} &= |(\lambda f)'_\alpha(a)| + |\lambda f|^0_{(\phi,2,\alpha)} \\ &= |\lambda|\,|f'_\alpha(a)| + |\lambda||f|^0_{(\phi,2,\alpha)} \end{aligned}$$
3. from Theorem 4.4 (I), $\quad = |\lambda||f|^R_{(\phi,2,\alpha)}, \lambda \in \mathbf{R}, f \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$.
$$\begin{aligned} |f+g|^R_{(\phi,2,\alpha)} &= |(f+g)'_\alpha(a)| + |f+g|^0_{(\phi,2,\alpha)} \\ &= |f'_\alpha(a) + g'_\alpha(a)| + |f+g|^0_{(\phi,2,\alpha)} \\ &\leq |f'_\alpha(a)| + |g'_\alpha(a)| + |f|^0_{(\phi,2,\alpha)} + |g|^0_{(\phi,2,\alpha)} \\ &= |f|^R_{(\phi,2,\alpha)} + |g|^R_{(\phi,2,\alpha)}, \quad f,g \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b]). \end{aligned}$$

4. .
If $|f|^R_{(\phi,2,\alpha)} = 0$, then $|f'_\alpha(a)| = 0$ and $|f|^0_{(\phi,2,\alpha)} = 0$, by Lemma 4.3 (III)
$$0 \leq V^R_{(\phi,2,\alpha)}(f) \leq |f|^0_{(\phi,2,\alpha)} = 0.$$
Thus $V^R_{(\phi,2,\alpha)}(f) = 0$, and from Theorem 3.4 (III)
$$f(x) = \lambda\alpha(x) + \mu, \quad \lambda, \mu \in \mathbf{R}.$$
Since $f'_\alpha(x) = \lambda$, $x \in [a, b]$ and $f'_\alpha(a) = 0$, we obtain $\lambda = 0$. Moreover, since $f \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a, b])$, $f(a) = 0$, therefore $\mu = 0$, hence $f = 0$.

□

Remark 4.7. *Let ϕ be a convex ϕ-function then*
$$\left(\mathbf{R}, RBV^0_{(\phi,2,\alpha)}([a, b]), +, |\cdot|^R_{(\phi,2,\alpha)}\right)$$
is a normed space.

Lemma 4.8. *Let ϕ be a ϕ-function. Then $f \in RBV_{(\phi,2,\alpha)}([a, b])$ if and only if $f - f(a) \in RBV^0_{(\phi,2,\alpha)}([a, b])$.*

Proof. From Definition 3.1 we observe that
$$\sigma^R_{(\phi,2,\alpha)}\left((f - f(a)), \Pi\right) = \sigma^R_{(\phi,2,\alpha)}\left(f, \Pi\right)$$
for all partition Π of $[a, b]$, which in turn implies that
$$V^R_{(\phi,2,\alpha)}(f - f(a)) = V^R_{(\phi,2,\alpha)}(f).$$
□

Now, by means of the norm $|\cdot|^R_{(\phi,2,\alpha)}$ on $\mathrm{RBV}^0_{(\phi,2,\alpha)}([a, b])$, we will define a norm $\|\cdot\|^R_{(\phi,2,\alpha)}$ on the vector space $\mathrm{RBV}_{(\phi,2,\alpha)}([a, b])$.

Definition 4.9. *Let ϕ be a convex ϕ-function.*
$$\begin{aligned} \|\cdot\|^R_{(\phi,2,\alpha)} : RBV_{(\phi,2,\alpha)}([a, b]) &\to [0, \infty) \\ f &\mapsto \|f\|^R_{(\phi,2,\alpha)} = |f(a)| + |f - f(a)|^R_{(\phi,2,\alpha)}. \end{aligned}$$

Next, we note that
$$\begin{aligned} \|f\|^R_{(\phi,2,\alpha)} &= |f(a)| + |f - f(a)|^R_{(\phi,2,\alpha)} \\ &= |f(a)| + |(f - f(a))'_\alpha(a)| + |f - f(a)|^0_{(\phi,2,\alpha)} \\ &= |f(a)| + |(f - f(a))'_\alpha(a)| + \inf\left\{\varepsilon > 0 : V^R_{(\phi,2,\alpha)}\left(\frac{f-f(a)}{\varepsilon}\right) \leq 1\right\} \\ &= |f(a)| + |f'_\alpha(a)| + \inf\left\{\varepsilon > 0 : V^R_{(\phi,2,\alpha)}\left(\frac{f}{\varepsilon}\right) \leq 1\right\}. \end{aligned}$$

Theorem 4.10. *Let ϕ be a convex ϕ-function. Then $\|\cdot\|^R_{(\phi,2,\alpha)}$ defines a norm on the linear space* $\Big(\mathbf{R}, RBV_{(\phi,2,\alpha)}([a,b]),+\Big)$.

Proof.

1. From the definition of $\|\cdot\|^R_{(\phi,2,\alpha)}$, it is clear that $\|\cdot\|^R_{(\phi,2,\alpha)} \geq 0$ for all $f \in \mathrm{RBV}_{(\phi,2,\alpha)}([a,b])$.

2. .
$$\begin{aligned}\|\lambda f\|^R_{(\phi,2,\alpha)} &= |\lambda f(a)| + |\lambda(f - f(a))|^R_{(\phi,2,\alpha)}\\ &= |\lambda||f(a)| + |\lambda||f - f(a)|^R_{(\phi,2,\alpha)}\end{aligned}$$
By Theorem 4.6 (II) and the fact that $f - f(a) \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$. Thus
$$\begin{aligned}\|\lambda f\|^R_{(\phi,2,\alpha)} &= |\lambda||f(a)| + |\lambda||f - f(a)|^R_{(\phi,2,\alpha)}\\ &= |\lambda|\left(|f(a)| + |f - f(a)|^R_{(\phi,2,\alpha)}\right) \quad \text{for } \lambda \in \mathbf{R},\, f \in \mathrm{RBV}_{(\phi,2,\alpha)}([a,b]).\\ &= |\lambda|\|f\|^R_{(\phi,2,\alpha)},\end{aligned}$$

3. Let $f, g \in \mathrm{RBV}_{(\phi,2,\alpha)}([a,b])$, then $f-f(a), g-g(a) \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$ and
$$\begin{aligned}\|f+g\|^R_{(\phi,2,\alpha)} &= |(f+g)(a)| + |(f+g) - (f+g)(a)|^R_{(\phi,2,\alpha)}\\ &= |f(a)+g(a)| + |(f - f(a)) + (g - g(a))|^R_{(\phi,2,\alpha)}\\ &\leq |f(a)| + |g(a)| + |f - f(a)|^R_{(\phi,2,\alpha)} + |g - g(a)|^R_{(\phi,2,\alpha)}\\ &= \|f\|^R_{(\phi,2,\alpha)} + \|g\|^R_{(\phi,2,\alpha)}.\end{aligned}$$

4. If $\|f\|^R_{(\phi,2,\alpha)} = 0$, then $|f(a)| = 0$ and $|f - f(a)|^R_{(\phi,2,\alpha)} = 0$, so $f(a) = 0$ and $f - f(a) = 0$, which implies that $f = f(a) = 0$.

□

Conclusion. Let ϕ be a convex ϕ function, then

1. $\Big(\mathbf{R}, \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b]), +, |\cdot|^R_{(\phi,2,\alpha)}\Big)$ is a normed space.
2. $\Big(\mathbf{R}, \mathrm{RBV}_{(\phi,2,\alpha)}([a,b]), +, \|\cdot\|^R_{(\phi,2,\alpha)}\Big)$ is a normed space.

5. Embedding results

We denote by
$$\mathrm{RV}^{(2,\alpha)^0}([a,b]) = \Big\{f : [a,b] \to \mathbf{R} : f \in \mathrm{RV}^{(2,\alpha)} \text{ and } f(a) = 0\Big\}$$

the linear space of all functions of bounded second α-variation that vanish at a. Our first embedding result states that $\mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$ is a subset of $\mathrm{RV}^{(2,\alpha)^0}([a,b])$.

Theorem 5.1. *Let ϕ be a convex ϕ-function. If $f \in RBV^0_{(\phi,2,\alpha)}([a,b])$, then $f \in RV^{(2,\alpha)^0}([a,b])$, and*

$$V^{(2,\alpha)}(f) \leq \left(\alpha(b) - \alpha(a) + \frac{1}{\phi(1)}\right) |f|^0_{(\phi,2,\alpha)}.$$

Therefore,

$$RBV^0_{(\phi,2,\alpha)}([a,b]) \hookrightarrow RV^{(2,\alpha)^0}([a,b]).$$

Proof. Let $f \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$. From Theorem 3.3, we know that $f \in \mathrm{RV}^{(2,\alpha)^0}([a,b])$.

If $|f|^R_{(\phi,2,\alpha)} = 0$, from Theorem 3.4 (III) we have $f(x) = \lambda\alpha(x) + \mu$, and therefore $V^{(2,\alpha)}(f) = 0$.

Next, let us consider f such that $|f|^R_{(\phi,2,\alpha)} \neq 0$, and

$$\Pi : a = x_{1,1} < x_{1,2} \leq x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \leq x_{2,3} < x_{2,4} = x_{3,1} < \cdots < x_{n-1,4} = x_{n,1} < x_{n,2} \leq x_{n,3} < x_{n,4} = b.$$

a partition of $[a,b]$. In the proof of Theorem 3.3 we change f by $f/|f|^0_{(\phi,2,\alpha)}$ in order to obtain

$$\begin{aligned} &\tfrac{1}{|f|^0_{(\phi,2,\alpha)}} \textstyle\sum_{j=1}^n \left| \tfrac{f(x_{j,4})-f(x_{j,3})}{\alpha(x_{j,4})-\alpha(x_{j,3})} - \tfrac{f(x_{j,2})-f(x_{j,1})}{\alpha(x_{j,2})-\alpha(x_{j,1})} \right| \\ &\leq \tfrac{1}{\phi(1)} V^R_{(\phi,2,\alpha)}\left(\tfrac{1}{|f|^0_{(\phi,2,\alpha)}}\right) + \alpha(b) - \alpha(a) \\ &\leq \tfrac{1}{\phi(1)} + \alpha(b) - \alpha(a), \quad \text{by Lemma 4.3 (I).} \end{aligned}$$

Thus

$$\sum_{j=1}^n \left| \frac{f(x_{j,4}) - f(x_{j,3})}{\alpha(x_{j,4}) - \alpha(x_{j,3})} - \frac{f(x_{j,2}) - f(x_{j,1})}{\alpha(x_{j,2}) - \alpha(x_{j,1})} \right| \leq \left(\frac{1}{\phi(1)} + \alpha(b) - \alpha(a)\right) |f|^0_{(\phi,2,\alpha)}.$$

Since this inequality holds for any partition Π of $[a,b]$, then

$$V^{(2,\alpha)}(f) \leq \left(\frac{1}{\phi(1)} + \alpha(b) - \alpha(a)\right) |f|^0_{(\phi,2,\alpha)}.$$

□

As a consequence of the previous result, we are able to compare the quantities $\|\cdot\|_{\mathrm{RV}^{(2,\alpha)}([a,b])}$ and $\|\cdot\|^R_{(\phi,2,\alpha)}$. More precisely, we have the following corollary.

Corollary 5.2. *Let ϕ be a convex ϕ-function. If $f \in RBV_{(\phi,2,\alpha)}([a,b])$, then $f \in RV^{(2,\alpha)}([a,b])$ and*

$$\|f\|_{\mathrm{RV}^{(2,\alpha)}([a,b])} \leq \max\left\{1, \alpha(b) - \alpha(a) + \frac{1}{\phi(1)}\right\} \|f\|^R_{(\phi,2,\alpha)}.$$

that is

$$RBV_{(\phi,2,\alpha)}([a,b]) \subset RV^{(2,\alpha)}([a,b]). \tag{5.1}$$

Proof. By Lemma 4.8, if $f \in \mathrm{RBV}_{(\phi,2,\alpha)}([a,b])$ then $f - f(a) \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$, and by Theorem 5.1, $f - f(a) \in \mathrm{RV}^{(2,\alpha)^0}([a,b])$. Note that

$$V^{(2,\alpha)}(f - f(a)) = V^{(2,\alpha)}(f).$$

One more time, from Theorem 5.1 we obtain

$$V^{(2,\alpha)}(f - f(a)) \leq \left(\frac{1}{\phi(1)} + \alpha(b) - \alpha(a)\right) |f - f(a)|^0_{(\phi,2,\alpha)},$$

then

$$\begin{aligned} &|f(a)| + |f'_\alpha(a)| + V^{(2,\alpha)}(f) \\ &\leq |f(a)| + |f'_\alpha(a)| + \left(\tfrac{1}{\phi(1)} + \alpha(b) - \alpha(a)\right) |f - f(a)|^0_{(\phi,2,\alpha)} \\ &\leq \max\left\{1, \tfrac{1}{\phi(1)} + \alpha(b) - \alpha(a)\right\} \left[|f(a)| + |f'_\alpha(a)| + |f - f(a)|^0_{(\phi,2,\alpha)}\right] \\ &= \max\left\{1, \tfrac{1}{\phi(1)} + \alpha(b) - \alpha(a)\right\} \|f\|^R_{(\phi,2,\alpha)}. \end{aligned}$$

Finally,

$$\|f\|_{\mathrm{RV}^{(2,\alpha)}([a,b])} \leq \max\left\{1, \frac{1}{\phi(1)} + \alpha(b) - \alpha(a)\right\} \|f\|^R_{(\phi,2,\alpha)}.$$

□

The ∞_1-condition is relevant here because the containment relationship (5.1) may be reversed, as the next theorem shows.

Theorem 5.3. *If ϕ is a convex ϕ-function which does not satisfy the ∞_1-condition, that is, there exists $r > 0$ such that*

$$\lim_{x\to\infty} \frac{\phi(x)}{x} = \sup_{x\in(0,\infty)} \frac{\phi(x)}{x} = r < \infty,$$

then

$$RV^{(2,\alpha)}([a,b]) \subset BV^R_{(\phi,2,\alpha)}([a,b]).$$

Moreover,

$$V_{(\phi,2,\alpha)}(f) \leq rV^{(2,\alpha)}(f), \quad f \in RV^{(2,\alpha)}([a,b]).$$

Proof. Let $f \in \mathrm{RV}^{(2,\alpha)}([a,b])$ and

$\Pi : a = x_{1,1} < x_{1,2} \leq x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \leq x_{2,3} < x_{2,4} = x_{3,1} < \cdots < x_{n-1,4} = x_{n,1} < x_{n,2} \leq x_{n,3} < x_{n,4} = b.$

a partition of $[a,b]$. Since ϕ satisfies the (∞_1)-condition, we obtain

$$\frac{\phi\left(\frac{|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right)}{\frac{|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}} \leq r, \quad j = 1, \ldots, n.$$

Hence $\phi\left(\frac{|f_\alpha[x_{j,4},x_{j,3}]-f_\alpha[x_{j,2},x_{j,1}]|}{|\alpha(x_{j,4})-\alpha(x_{j,1})|}\right) |\alpha(x_{j,4}) - \alpha(x_{j,1})|$
$\leq r \left|f_\alpha[x_{j,4}, x_{j,3}] - f_\alpha[x_{j,2}, x_{j,1}]\right|, \quad j = 1, \ldots, n.$

From this last inequality we deduce the following

$$\sigma^R_{(\phi,2,\alpha)}(f,\Pi) \leq r \sum_{j=1}^{n} \left| \frac{f(x_{j,4}) - f(x_{j,3})}{\alpha(x_{j,4}) - \alpha(x_{j,3})} - \frac{f(x_{j,2}) - f(x_{j,1})}{\alpha(x_{j,2}) - \alpha(x_{j,1})} \right|$$

and thus

$$\sigma^R_{(\phi,2,\alpha)}(f,\Pi) \leq rV^{(2,\alpha)}(f),$$

for all partition Π of $[a,b]$. Therefore,

$$V^R_{(\phi,2,\alpha)}(f) \leq rV^{(2,\alpha)}(f),$$

which means that $f \in \mathrm{BV}^R_{(\phi,2,\alpha)}([a,b])$. □

Actually, for a function which does not satisfy the ∞_1-condition, we can combine Theorem 5.3 and Theorem 3.3 and conclude that, in this case, the $(2,\alpha)$-variation is equivalent to the $(\phi,2,\alpha)$-variation, as shown in the corollary below.

Corollary 5.4. *Let ϕ be a convex ϕ-function which does not satisfy the ∞_1-condition. Then*

$$RV^{(2,\alpha)}([a,b]) = V^R_{(\phi,2,\alpha)}([a,b]),$$

and

$$\frac{1}{r}V^R_{(\phi,2,\alpha)}(f) \leq V^{(2,\alpha)}(f) \leq \alpha(b) - \alpha(a) + \frac{1}{\phi(1)}V^R_{(\phi,2,\alpha)}(f).$$

The following are the embedding results for functions which satisfy the ∞_1-condition. Compare with Equation (42) in [3].

Corollary 5.5. *If ϕ is a convex ϕ-function such that satisfies the ∞_1-condition, then we have the following embedding results*

$$RBV_{(\phi,2,\alpha)}([a,b]) \subset RV^{(2,\alpha)}([a,b]) \subset \alpha - Lip([a,b]) \subset RV_{(\phi,\alpha)} \subset \cdots$$
$$\subset \alpha - AC([a,b]) \subset BV([a,b]) \subset B([a,b]).$$

6. RBV$(\phi, 2, \alpha)([a, b])$ as a Banach space

First of all, consider the case of the $(\phi, 2, \alpha)$-bounded variation functions which vanish at a.

Theorem 6.1. *If ϕ is a convex ϕ-function which satisfies the ∞_1-condition, then*

$$\left(\mathbf{R}, RBV^0_{(\phi,2,\alpha)}([a, b]), +, |\cdot|^R_{(\phi,2,\alpha)}\right)$$

is a complete space.

Proof. Let $\{f_n\}_{n\in N}$ be a Cauchy sequence in $\mathrm{RBV}^0_{(\phi,2,\alpha)}([a, b])$. Given $\varepsilon > 0$, there exists $N_\varepsilon \in N$ such that $q, r > N_\varepsilon$ implies that $|f_q - f_r|^R_{(\phi,2,\alpha)}$. From this last inequality, we deduce

$$\begin{cases} |(f_q)'_\alpha(a) - (f_r)'_\alpha(a)| < \varepsilon \\ |f_q - f_r|^0_{(\phi,2,\alpha)} < \varepsilon. \end{cases}$$

By Corollary 5.5 we obtain

$$\mathrm{RV}^0_{(\phi,2,\alpha)}([a, b]) \subset B^0([a, b]).$$

($B^0([a, b])$ is the Banach space of all bounded functions on $[a, b]$ vanishing at a). Thus, there exists $k > 0$ such that

$$\|f_q - f_r\|_\infty \leq k\,|f_q - f_r|^R_{(\phi,2,\alpha)} < k\varepsilon,$$

which implies that $\{f_k\}_{k\in N}$ is a Cauchy sequence in $\left(B^0([a, b]), \|\cdot\|_\infty\right)$ and therefore converges uniformly to a function $f \in B^0([a, b]$. We can define $f : [a, b] \to \mathbf{R}$ by $x \mapsto f(x) = \lim_{k\to\infty} f_k(x)$. We have to prove that f satisfies the following conditions:

1. $f \in \mathrm{RV}^0_{(\phi,2,\alpha)}([a, b])$.
2. $\{f_k\}_{k\in N}$ converges in the norm $|\cdot|^R_{(\phi,2,\alpha)}$.

Since $|f_q - f_r|^0_{(\phi,2,\alpha)} < \varepsilon$, from Lemma 4.3 (II) we deduce that

$$V^R_{(\phi,2,\alpha)}\left(\frac{f_q - f_r}{\varepsilon}\right) \leq 1.$$

Let

$\Pi : a = x_{1,1} < x_{1,2} \leq x_{1,3} < x_{1,4} = x_{2,1} < x_{2,2} \leq x_{2,3} < x_{2,4} = x_{3,1} < \cdots < x_{n-1,4} = x_{n,1} < x_{n,2} \leq x_{n,3} < x_{n,4} = b.$

be a partition of $[a, b]$. Then

$$\begin{aligned}\sigma^R_{(\phi,2,\alpha)}\left(\tfrac{f_q - f}{\varepsilon}, \Pi\right) &= \sigma^R_{(\phi,2,\alpha)}\left(\tfrac{f_q - \lim_{r\to\infty} f_r}{\varepsilon}, \Pi\right) \\ &= \lim_{r\to\infty} \sigma^R_{(\phi,2,\alpha)}\left(\tfrac{f_q - f_r}{\varepsilon}, \Pi\right).\end{aligned}$$

Given that ϕ is a continuous function and we are considering finite sums, then we have

$$\sigma^R_{(\phi,2,\alpha)}\left(\frac{f_q - f}{\varepsilon}, \Pi\right) \leq \lim_{r\to\infty} V^R_{(\phi,2,\alpha)}\left(\frac{f_q - f_r}{\varepsilon}\right) \leq 1.$$

This holds for all partition Π of $[a, b]$. Hence

$$V^R_{(\phi,2,\alpha)}\left(\frac{f_q - f}{\varepsilon}\right) \leq 1, \quad \text{if } q > N_\varepsilon.$$

And so

$$f_q - f \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a, b]).$$

Given the $\mathrm{RBV}^0_{(\phi,2,\alpha)}([a, b])$ is a linear space and $f_q \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a, b])$, it is concluded that $f \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a, b])$. This proves i).

This fact guarantees the existence of $f'_\alpha(a)$ $(= f'_{\alpha^+}(a))$.

Now, le us see that the sequence $\{f_n\}_{n\in N}$ converges to f in the norm $|\cdot|^R_{(\phi,2,\alpha)}$. In order to do that we use the following result:

$$V^R_{(\phi,2,\alpha)}\left(\frac{f_q - f}{\varepsilon}\right) \leq 1, \quad \text{if } q > N_\varepsilon.$$

By Lemma 4.3 (II) we have

$$|f_q - f|^0_{(\phi,2,\alpha)} < \varepsilon, \quad \text{if } q > N_\varepsilon.$$

Moreover

$$|(f_q)'_\alpha(a) - (f_r)'_\alpha(a)| < \varepsilon, \quad \text{if } q, r > N_\varepsilon.$$

Then

$$\left|(f_q)'_\alpha(a) - \lim_{h\to 0} \frac{f_r(a+h) - f_r(a)}{\alpha(a+h) - \alpha(a)}\right| < \varepsilon, \quad \text{if } q, r > N_\varepsilon.$$

And so

$$\lim_{r\to\infty}\left|(f_q)'_\alpha(a) - \lim_{h\to 0} \frac{f_r(a+h) - f_r(a)}{\alpha(a+h) - \alpha(a)}\right| < \varepsilon, \quad \text{if } q, r > N_\varepsilon.$$

Since the convergence of $\{f_k\}_{k\in N}$ is uniform, we obtain

$$\left|(f_q)'_\alpha(a) - \lim_{h\to 0}\frac{f_r(a+h)-f_r(a)}{\alpha(a+h)-\alpha(a)}\right| < \varepsilon, \quad \text{if } q > N_\varepsilon.$$

This is

$$|(f_q)'_\alpha(a) - f'_\alpha(a)| < \varepsilon, \quad \text{if } q > N_\varepsilon.$$

Finally, for $q > N_\varepsilon$ we have the following

$$|f_q - f|^R_{(\phi,2,\alpha)} = |(f_q)'_\alpha(a) - f'_\alpha(a)| + |f_q - f|^0 < 2\varepsilon.$$

Which tells us that the sequence $\{f_k\}_{k\in N}$ converges to $f \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$ in the norm $|\cdot|^R_{(\phi,2,\alpha)}$. $\square$

Now, by using the previous result, we can prove the completeness of the space $\mathrm{RBV}_{(\phi,2,\alpha)}([a,b])$.

Theorem 6.2. *If ϕ is a convex ϕ-function which satisfies the ∞_1-condition, then*

$$\left(\mathbf{R}, RBV_{(\phi,2,\alpha)}([a,b]), +, \|\cdot\|^R_{(\phi,2,\alpha)}\right)$$

is a complete space.

Proof. Let $\{f_n\}_{n\in N}$ be a Cauchy sequence in $\mathrm{RBV}_{(\phi,2,\alpha)}([a,b])$. Given $\varepsilon > 0$ there exists $N_\varepsilon \in N$ such that

$$\|f_p - f_q\|^R_{(\phi,2,\alpha)} < \varepsilon, \quad \text{if } p, q > N_\varepsilon.$$

Which means that

$$|(f_p - f_q)(a)| + |(f_p - f_q) - (f_p - f_q)(a)|^R_{(\phi,2,\alpha)} < \varepsilon.$$

Let $g_n = f_n - f$, $n \in N$, then by Lemma 4.8 we have $g_n \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$. Hence

$$|g_p - g_q|^R_{(\phi,2,\alpha)} < \varepsilon$$

which implies that $\{g_n\}_{n\in N}$ is a Cauchy sequence in $\left(\mathbf{R}, \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b]), +, |\cdot|^R_{(\phi,2,\alpha)}\right)$ which is a complete space. Therefore $\{g_n\}_{n\in N}$ converges in the norm $|\cdot|^R_{(\phi,2,\alpha)}$ to a function $g \in \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b])$.

Since $|(f_p - f_q)(a)| < \varepsilon$ for $p, q > N_\varepsilon$ we have that the sequence $\{f_n(a)\}_{n\in N}$ is a Cauchy sequence in $\mathbf{R}$ and so it converges to $f_0 \in \mathbf{R}$.

Let $f = g + f_0$. Then $f \in \mathrm{RBV}_{(\phi,2,\alpha)}([a,b])$, since g and f_0 have $(\phi,2,\alpha)$-bounded variation in the sense of Riesz and

$$f(a) = (g + f_0)(a) = g(a) + f_0 = f_0$$

thus $g = f - f(a)$. Moreover,

$$\begin{aligned}\|f_n - f\|^R_{(\phi,2,\alpha)} &= |(f_n - f)(a)| + |(f_n - f) - (f_n - f)(a)|^R_{(\phi,2,\alpha)}\\ &= |f_n(a) - f(a)| + |g_n - g|^R_{(\phi,2,\alpha)}.\end{aligned}$$

Now, since $f_n(a) \underset{n\to\infty}{\longrightarrow} f_0$ and $g_n \underset{n\to\infty}{\longrightarrow} g$ in the norm $|\cdot|^R_{(\phi,2,\alpha)}$, we have

$$f_n \underset{n\to\infty}{\longrightarrow} f \in \mathrm{RBV}_{(\phi,2,\alpha)}([a,b])$$

in the norm $\|\cdot\|^R_{(\phi,2,\alpha)}$. □

Conclusion.

1. $\left(\mathbf{R}, \mathrm{RBV}^0_{(\phi,2,\alpha)}([a,b]), +, |\cdot|^R_{(\phi,2,\alpha)}\right)$ is a Banach space.
2. $\left(\mathbf{R}, \mathrm{RBV}_{(\phi,2,\alpha)}([a,b]), +, \|\cdot\|^R_{(\phi,2,\alpha)}\right)$ is a Banach space.

Acknowledgments.

H. C. Chaparro was supported by Research Office- UMNG through the project INV-CIAS-3151.

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