On generalizations of graded second submodules

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Abstract:

Let \(G\) be a group with identity \(e\), \(R\) be a commutative \(G\)-graded ring with unity \(1\) and \(M\) be a \(G\)-graded \(R\)-module. In this article, we introduce and study two generalizations of graded second submodules, namely, graded 2-absorbing second submodules and graded strongly 2-absorbing second submodules. Also, we introduce and study the concept of graded quasi 2-absorbing second submodules, that is a generalization for graded strongly 2-absorbing second submodules.

Keywords: Graded second submodules; Graded 2-absorbing submodules; Graded strongly 2-absorbing submodules; Graded 2-absorbing second submodules; Graded strongly 2-absorbing second submodules; Graded quasi 2-absorbing second submodules.

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1. Introduction

Throughout this article, $G$ will be a group with identity $e$ and $R$ will be a commutative ring with a nonzero unity $1$. $R$ is said to be $G$-graded if $R = \bigoplus_{g \in G} R_g$ with $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$ where $R_g$ is an additive subgroup of $R$ for all $g \in G$. The elements of $R_g$ are called homogeneous of degree $g$. Consider $\text{supp}(R, G) = \{g \in G : R_g = 0\}$. If $x \in R$, then $x$ can be written as $\sum_{g \in G} x_g$, where $x_g$ is the component of $x$ in $R_g$. Also, $h(R) = \bigcup_{g \in G} R_g$. Moreover, it has been proved in [21] tha $R_e$ is a subring of $R$ and $1 \in R_e$.

Let $I$ be an ideal of a graded ring $R$. Then $I$ is said to be graded ideal if $I = \bigoplus_{g \in G} (I \cap R_g)$, i.e., for $x \in I$, $x = \sum_{g \in G} x_g$ where $x_g \in I$ for all $g \in G$. Let $R$ be a $G$-graded ring and $I$ be a graded ideal of $R$. Then $R/I$ is $G$-graded by $(R/I)_g = (R_g + I)/I$ for all $g \in G$.

Assume that $M$ is a left $R$-module. Then $M$ is said to be $G$-graded if $M = \bigoplus_{g \in G} M_g$ with $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$ where $M_g$ is an additive subgroup of $M$ for all $g \in G$. The elements of $M_g$ are called homogeneous of degree $g$. Also, we consider $\text{supp}(M, G) = \{g \in G : M_g = 0\}$. It is clear that $M_g$ is an $R_e$-submodule of $M$ for all $g \in G$. Moreover, $h(M) = \bigcup_{g \in G} M_g$.

Let $N$ be an $R$-submodule of a graded $R$-module $M$. Then $N$ is said to be graded $R$-submodule if $N = \bigoplus_{g \in G} (N \cap M_g)$, i.e., for $x \in N$, $x = \sum_{g \in G} x_g$ where $x_g \in N$ for all $g \in G$. Let $M$ be a $G$-graded $R$-module and $N$ be a graded $R$-submodule of $M$. Then $M/N$ is a graded $R$-module by $(M/N)_g = (M_g + N)/N$ for all $g \in G$.

Lemma 1.1. ([16]) Let $R$ be a $G$-graded ring and $M$ be a $G$-graded $R$-module.

1. If $I$ and $J$ are graded ideals of $R$, then $I + J$ and $I \cap J$ are graded ideals of $R$.

2. If $N$ and $K$ are graded $R$-submodules of $M$, then $N + K$ and $N \cap K$ are graded $R$-submodules of $M$. 
3. If \( N \) is a graded \( R \)-submodule of \( M \), \( r \in h(R) \), \( x \in h(M) \) and \( I \) is a graded ideal of \( R \), then \( Rx, IN \) and \( rN \) are graded \( R \)-submodules of \( M \). Moreover, \((N :_R M) = \{ r \in R : rM \subseteq N \}\) is a graded ideal of \( R \).

Also, it has been proved in [17] that if \( N \) is a graded \( R \)-submodule of \( M \), then \( \text{Ann}_R(N) = \{ r \in R : rN = \{0\} \}\) is a graded ideal of \( R \).

Graded prime submodules have been introduced by Atani in [12]. A proper graded \( R \)-submodule \( N \) of \( M \) is said to be graded prime if whenever \( r \in h(R) \) and \( m \in h(M) \) such that \( rm \in N \), then either \( m \in N \) or \( r \in (N :_R M) \). Graded prime submodules have been widely studied by several authors, for more details one can look in [1], [2], [4] and [8].

Let \( M \) and \( S \) be two \( G \)-graded \( R \)-modules. An \( R \)-homomorphism \( f : M \to S \) is said to be graded \( R \)-homomorphism if \( f(M_g) \subseteq S_g \) for all \( g \in G \) (see [21]). Graded second submodules have been introduced by Ansari-Toroghy and Farshadifar in [9]. A nonzero graded \( R \)-submodule \( N \) of \( M \) is said to be graded second if for each \( a \in h(R) \), the graded \( R \)-homomorphism \( f : N \to N \) defined by \( f(x) = ax \) is either surjective or zero. In this case, \( \text{Ann}_R(N) \) is a graded prime ideal of \( R \). Graded second submodules have been wonderfully studied by Çeken and Alkan in [14]. On the other hand, graded secondary modules have been introduced by Atani and Farzalipour in [13]. A nonzero graded \( R \)-module \( M \) is said to be graded secondary if for each \( a \in h(R) \), the graded \( R \)-homomorphism \( f : M \to M \) defined by \( f(x) = ax \) is either surjective or nilpotent.

In [20], Naghani and Moghimi gave a generalization of graded prime ideals, called graded 2-absorbing ideals. A proper graded ideal \( P \) of \( R \) is said to be graded 2-absorbing if whenever \( a, b, c \in h(R) \) such that \( abc \in P \), then either \( ab \in P \) or \( ac \in P \) or \( bc \in P \). Graded 2-absorbing ideals have been admirably studied in [6].

The authors in [5] extended graded 2-absorbing ideals to graded 2-absorbing submodules. A proper graded \( R \)-submodule \( N \) of \( M \) is said to be graded 2-absorbing if whenever \( a, b \in h(R) \) and \( m \in h(M) \) such that \( abm \in N \), then either \( am \in N \) or \( bm \in N \) or \( ab \in (N :_R M) \). Graded 2-absorbing submodules have been deeply studied in [7].

In [15], a proper \( \mathbb{Z} \)-graded \( R \)-submodule \( N \) of \( M \) is said to be graded completely irreducible if whenever \( N = \bigcap_{k \in \Delta} N_k \) where \( \{N_k\}_{k \in \Delta} \) is a family of \( \mathbb{Z} \)-graded \( R \)-submodules of \( M \), then \( N = N_k \) for some \( k \in \Delta \). In
[19], the concept of graded completely irreducible submodules has been extended into $G$-graded case, for any group $G$. It has been proved that every graded $R$-submodule of $M$ is an intersection of graded completely irreducible $R$-submodules of $M$. In many instances, we use the following basic fact without further discussion.

**Remark 1.2.** Let $N$ and $L$ be two graded $R$-submodules of $M$. To prove that $N \subseteq L$, it is enough to prove that: If $K$ is a graded completely irreducible $R$-submodule of $M$ such that $L \subseteq K$, then $N \subseteq K$.

The purpose of our article is to follow [11] in order to introduce and study the concept of graded 2-absorbing second submodules, that is a generalization of graded second submodules. A nonzero graded $R$-submodule $N$ of $M$ is said to be graded 2-absorbing second if whenever $x, y \in h(R)$ and $K$ is a graded completely irreducible $R$-submodule of $M$ such that $xyN \subseteq K$, then either $xN \subseteq K$ or $yN \subseteq K$ or $xy \in \text{Ann}_R(N)$. Also, we follow [11] to introduce another generalization, namely, graded strongly 2-absorbing second submodules. A nonzero graded $R$-submodule $N$ of $M$ is said to be graded strongly 2-absorbing second if whenever $x, y \in h(R)$ and $K$ is a graded $R$-submodule of $M$ such that $xyN \subseteq K$, then either $xN \subseteq K$ or $yN \subseteq K$ or $xy \in \text{Ann}_R(N)$.

In Corollary 3.4, we prove that if $N$ is a graded strongly 2-absorbing second $R$-submodule of $M$, then $\text{Ann}_R(N)$ is a graded 2-absorbing ideal of $R$, and in Example 3.5, we show that the converse is not true in general. Motivated by this, we introduce and study a generalization for graded strongly 2-absorbing second submodules. A nonzero graded $R$-submodule $N$ of $M$ is said to be graded quasi 2-absorbing second if $\text{Ann}(N)$ is a graded 2-absorbing ideal of $R$. Related results have been obtained.

2. Graded 2-Absorbing Second Submodules

In this section, we introduce and study the concept of graded 2-absorbing second submodules.

**Definition 2.1.** Let $M$ be a graded $R$-module and $N$ be a nonzero graded $R$-submodule of $M$. Then $N$ is said to be a graded 2-absorbing second $R$-submodule of $M$ if whenever $x, y \in h(R)$ and $K$ is a graded completely irreducible $R$-submodule of $M$ such that $xyN \subseteq K$, then either $xN \subseteq K$ or $yN \subseteq K$ or $xy \in \text{Ann}_R(N)$. 
Example 2.2. Let $R = \mathbb{Z}$, $M = \mathbb{Z}_n[i]$ and $G = \mathbb{Z}_4$. Then $R$ is $G$-graded by $R_0 = \mathbb{Z}$ and $R_1 = R_2 = R_3 = \{0\}$. Also, $M$ is $G$-graded by $M_0 = \mathbb{Z}_n$, $M_2 = i\mathbb{Z}_n$ and $M_1 = M_3 = \{0\}$. Consider the graded $R$-submodule $N = \mathbb{Z}_n$ of $M$. If $n = p$ or $n = pq$ where $p$, $q$ are primes, then $N$ is a graded $2$-absorbing second $R$-submodule of $M$.

Example 2.3. Let $R = \mathbb{Z}$, $M = \mathbb{Z}[i]$ and $G = \mathbb{Z}_2$. Then $R$ is $G$-graded by $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Also, $M$ is $G$-graded by $M_0 = \mathbb{Z}$ and $M_1 = i\mathbb{Z}$. Consider the graded $R$-submodule $N = n\mathbb{Z}$ of $M$. Obviously, $n = p_1^{r_1}p_2^{r_2}....p_k^{r_k}$ where $p_i^{r_i}$ $(1 \leq i \leq k)$ are distinct primes. Now, $p_1 \in h(R)$ and $K = p_1^{r_1+1}Z$ is a graded completely irreducible $R$-submodule of $M$ such that $p_1 p_1 N \subseteq K$. But $p_1 N \subseteq K$ and $p_1 p_1 \notin \text{Ann}_R(N) = \{0\}$. Hence, $N$ is not graded $2$-absorbing second $R$-submodule of $M$.

Remark 2.4. Consider the $\mathbb{Z}$-module $\mathbb{Z}$ and assume it is $G$-graded by any group $G$. Since the only graded submodules are $n\mathbb{Z}$, then by Example 2.3, $\mathbb{Z}$ has no graded $2$-absorbing second submodules.

Let $\Omega(M)$ be the set of all graded completely irreducible $R$-submodules of $M$. Assume that $P$ is a graded prime ideal of $R$ and $N$ is a graded $R$-submodule of $M$. Then we define

$$I_P^M(N) = \bigcap_{K \in \Omega(M)} \{K : rN \subseteq K \text{ for some } r \in h(R) - P\}$$

. The following lemma gives some characterizations for graded second $R$-submodules.

Lemma 2.5. Let $N$ be a graded $R$-submodule of a graded $R$-module $M$. Then the following are equivalent.

1. If $N = \{0\}$, $K$ is a graded completely irreducible $R$-submodule of $M$ and $r \in h(R)$ such that $rN \subseteq K$, then either $rN = \{0\}$ or $N \subseteq K$.

2. $N$ is a graded second $R$-submodule of $M$.

3. $P = \text{Ann}_R(N)$ is a graded prime ideal of $R$ and $I_P^M(N) = N$.

Proof. (1) $\Rightarrow$ (2): Suppose that $r \in h(R)$ and $rN = \{0\}$. If $rN \subseteq K$ for some graded completely irreducible $R$-submodule $K$ of $M$, then by assumption, $N \subseteq K$. Hence, $N \subseteq rN$. (2) $\Rightarrow$ (3): By [9], $P = \text{Ann}_R(N)$ is a graded prime ideal of $R$. Now, let $K$ be a graded completely irreducible
$R$-submodule of $M$ and $r \in h(R) - P$ such that $rN \subseteq K$. Then $N \subseteq K$
by assumption. Therefore, $N \subseteq I^M_R(N)$. The reverse inclusion is clear.
$(3) \Rightarrow (1)$: Since $Ann_R(N)$ is a graded prime ideal of $R$, $N = \{0\}$. Let $K$
be a graded completely irreducible $R$-submodule of $M$ and $r \in h(R)$ such
that $rN \subseteq K$. Suppose that $rN = \{0\}$. Then $r \in h(R) - P$. Therefore,
$I^M_R(N) = N$ by assumption. Hence, $N \subseteq K$, as desired. □

**Proposition 2.6.** Let $M$ be a graded $R$-module. If either $L$ is a graded
second $R$-submodule of $M$ or $L$ is a sum of two graded second $R$-submodules
of $M$, then $L$ is a graded 2-absorbing second $R$-submodule of $M$.

**Proof.** The first assertion is clear. Let $N$ and $L$ be two graded second
$R$-submodules of $M$. We show that $N + L$ is a graded 2-absorbing second $R$-
submodules of $M$. Let $x, y \in h(R)$ and $K$ is a graded completely irreducible
$R$-submodule of $M$ such that $xy(N + L) \subseteq K$. Since $N$ is graded second,
either $xy = \{0\}$ or $N \subseteq K$ by Lemma 2.5. Similarly, either $xyL = \{0\}$
or $L \subseteq K$. If $xyN = xyL = \{0\}$, then we are done. Also, if $N \subseteq K$
and $L \subseteq K$, then we are done. Assume that $xyN = \{0\}$ and $L \subseteq K$. Then
$xN = \{0\}$ or $yN = \{0\}$ because $Ann_R(N)$ is a graded prime ideal of $R$. If
$xN = \{0\}$, then $x(N + L) \subseteq xN + L \subseteq L \subseteq K$. Similarly, if $yN = \{0\}$, we
have $y(N + L) \subseteq L$ as desired. □

**Proposition 2.7.** Let $M$ be a graded $R$-module. If $L$ is a graded sec-
ondary $R$-submodule of $M$ and $R/Ann_R(L)$ has no nonzero nilpotent homoge-
neous element, then $L$ is a graded 2-absorbing second $R$-submodule
of $M$.

**Proof.** Let $x, y \in h(R)$ and $K$ be a graded completely irreducible $R$-
submodule of $M$ such that $xyL \subseteq K$. If $xL \subseteq K$ or $yL \subseteq K$, then we are
done. Suppose that $xL \subseteq K$ and $yL \subseteq K$. Then $x, y \in R/Ann_R(L)$. Thus,
$(xy)^r \in Ann_R(L)$ for some positive integer $r$. Therefore, $xy \in Ann_R(L)$
since $R/Ann_R(L)$ has no nonzero nilpotent homogeneous element. Hence,
$L$ is a graded 2-absorbing second $R$-submodule of $M$. □

**Proposition 2.8.** Let $M$ be a $G$-graded $R$-module, $I$ be a graded ideal of
$R$ and $L$ be a graded 2-absorbing second $R$-submodule of $M$. If $x \in h(R)$
and $K$ is a graded completely irreducible $R$-submodule of $M$ such that
$xIL \subseteq K$, then either $xL \subseteq K$ or $xI \subseteq Ann_R(L)$ or $I_pL \subseteq K$ for some
g $\in G$. 
Proof. Suppose that \( xL \subseteq K \) and \( xI \subseteq \text{Ann}_R(L) \). Then there exists \( y \in I \) such that \( xyL = \{0\} \), and then there exists \( g \in G \) such that \( xygL = \{0\} \) where \( yg \in I \) since \( I \) is graded. Now, since \( L \) is graded 2-absorbing second and \( xygL \subseteq K \), we have \( ygL \subseteq K \). We show that \( I_yL \subseteq K \).

Let \( z_g \in I_y \). Then \( (yg + zg)xL \subseteq K \). Hence, either \( (yg + zg)L \subseteq L \) or \( (yg + zg)x \in \text{Ann}_R(L) \). If \( (yg + zg)L \subseteq K \), then since \( ygL \subseteq K \), we have \( zgL \subseteq K \). If \( (yg + zg)x \in \text{Ann}_R(L) \), then \( zg \not\in \text{Ann}_R(L) \), but \( zg_2L \subseteq K \).

Thus, \( zgL \subseteq K \). Hence, we conclude that \( I_yL \subseteq K \). \( \Box \)

Lemma 2.9. Let \( M \) be a \( G \)-graded \( R \)-module and \( N \) a graded \( R \)-submodule of \( M \). If \( r \in h(R) \), then \( (N :_M r) = \{ m \in M : rm \in N \} \) is a graded \( R \)-submodule of \( M \).

Proof. Clearly, \( (N :_M r) \) is a graded \( R \)-submodule of \( M \). Let \( m \in (N :_M r) \). Then \( rm \in N \). Now, \( m = \sum_{g \in G} m_g \) where \( m_g \in M_g \) for all \( g \in G \).

Since \( r \in h(R) \), \( r \in R_h \) for some \( h \in G \) and then \( rm_g \in M_{hg} \subseteq h(M) \) for all \( g \in G \) such that \( \sum_{g \in G} rm_g = r \left( \sum_{g \in G} m_g \right) = rm \in N \). Since \( N \) is graded, \( rm_g \in N \) for all \( g \in G \) which implies that \( m_g \in (N :_M r) \) for all \( g \in G \). Hence, \( (N :_M r) \) is a graded \( R \)-submodule of \( M \). \( \Box \)

In [23], a graded \( R \)-module \( M \) is said to be graded cocyclic if the sum of all graded minimal \( R \)-submodules of \( M \) is a large and graded simple \( R \)-submodule of \( M \).

Lemma 2.10. A graded \( R \)-submodule \( K \) of \( M \) is a graded completely irreducible \( R \)-submodule of \( M \) if and only if \( M/K \) is a graded cocyclic \( R \)-module.

Proof. It follows from ([18], Remark 1.1). \( \Box \)

Lemma 2.11. Let \( K \) be a graded completely irreducible \( R \)-submodule of \( M \). Then \( (K :_M r) \) is a graded completely irreducible \( R \)-submodule of \( M \) for all \( r \in h(R) \).

Proof. This follows from Lemma 2.9, Lemma 2.10 and that \( M/(K :_M r) \cong (rM + K)/K \). \( \Box \)

Proposition 2.12. Let \( L \) be a graded 2-absorbing second \( R \)-submodule of \( M \) and \( K \) is a graded completely irreducible \( R \)-submodule of \( M \) such that \( L \subseteq K \), Then \( (K :_R L) \) is a graded 2-absorbing ideal of \( R \).
Proof. Since $L \subseteq K$, we have $(K:_R L) = R$. Let $x, y, z \in h(R)$ such that $xyz \in (K:_R L)$. Then $xyL \in (K:_L z)$. Thus $xL \subseteq (K:_M z)$ or $yL \subseteq (K:_M z)$ or $xyL = \{0\}$ since $L$ is graded 2-absorbing second and $(K:_M z)$ is a graded completely irreducible $R$-submodule of $M$ by Lemma 2.11. Therefore, $xz \in (K:_R L)$ or $yz \in (K:_R L)$ or $xy \in (K:_R L)$. Hence, $(K:_R L)$ is a graded 2-absorbing ideal of $R$. \qed

Corollary 2.13. If $M$ is a graded cocyclic $R$-module and $L$ is a graded 2-absorbing second $R$-submodule of $M$, then $\text{Ann}_R(L)$ is a graded 2-absorbing ideal of $R$.

Proof. Since $M$ is graded cocyclic, $\{0\}$ is a graded completely irreducible $R$-submodule of $M$ by Lemma 2.10. Thus the result follows from Proposition 2.12. \qed

Proposition 2.14. Let $L$ be a graded 2-absorbing second $R$-submodule of $M$. Then $x^nL = x^{n+1}L$ for all $x \in h(R)$ and $n \geq 2$.

Proof. It is enough to prove that $x^2L = x^3L$. Let $x \in h(R)$. Then clearly, $x^3L \subseteq x^2L$. Let $K$ be a graded completely irreducible $R$-submodule of $M$ such that $x^3L \subseteq K$. Then $x^2L \subseteq (K:_M x)$. Thus $xL \subseteq (K:_M x)$ or $x^2L = \{0\}$ since $L$ is graded 2-absorbing second submodule of $M$ and $(K:_M x)$ is a graded completely irreducible $R$-submodule of $M$ by Lemma 2.11. Therefore, $x^2L \subseteq K$. Hence, $x^2L = x^3L$. \qed

Proposition 2.15. Let $L$ be a graded 2-absorbing second $R$-submodule of $M$. If $\text{Ann}_R(L)$ is a graded prime ideal of $R$, then $(K:_R L)$ is a graded prime ideal of $R$ for all graded completely irreducible $R$-submodule $K$ of $M$ with $L \subseteq K$.

Proof. Let $K$ be a graded completely irreducible $R$-submodule of $M$ such that $L \subseteq K$. Assume that $x, y \in h(R)$ such that $xy \in (K:_R L)$. Then $xyL \subseteq K$; and then $xL \subseteq K$ or $yL \subseteq K$ or $xyL = \{0\}$. If $xyL = \{0\}$, then $xL = \{0\}$ or $yL = \{0\}$. So, in all cases, we have $xL \subseteq K$ or $yL \subseteq K$, which implies that $x \in (K:_R L)$ or $y \in (K:_R L)$. Hence, $(K:_R L)$ is a graded prime ideal of $R$. \qed

Proposition 2.16. Let $L$ be a graded 2-absorbing second $R$-submodule of $M$. If $\text{Grad}(\text{Ann}_R(L)) = P$ for some graded prime ideal $P$ of $R$ and $K$ is a graded completely irreducible $R$-submodule of $M$ such that $L \subseteq K$, then $\text{Grad}((K:_R L))$ is a graded prime ideal of $R$ containing $P$. 
Proof. Let \( x, y \in h(R) \) such that \( xy \in \text{Grad}(K :_R L) \). Then \( x^r y^r L \subseteq K \) for some positive integer \( r \), and then \( x^r L \subseteq K \) or \( y^r L \subseteq K \) or \( x^r y^r L = \{0\} \). If \( x^r L \subseteq K \) or \( y^r L \subseteq K \), then we are done. Suppose that \( x^r y^r L = \{0\} \). Then \( xy \in \text{Grad}(\text{Ann}_R(L)) = P \). Thus \( x \in P \) or \( y \in P \). Clearly, \( P = \text{Grad}(\text{Ann}_R(L)) \subseteq \text{Grad}(K :_R L) \). Therefore, \( x \in \text{Grad}(K :_R L) \) or \( y \in \text{Grad}(K :_R L) \). □

3. Graded Strongly 2-Absorbing Second Submodules

In this section, we introduce and study the concept of graded strongly 2-absorbing second submodules.

Definition 3.1. Let \( M \) be a graded \( R \)-module. Then a nonzero graded \( R \)-submodule \( N \) of \( M \) is said to be graded strongly 2-absorbing second if whenever \( x, y \in h(R) \) and \( K \) is a graded \( R \)-submodule of \( M \) such that \( xyN \subseteq K \), then either \( xN \subseteq K \) or \( yN \subseteq K \) or \( xy \in \text{Ann}_R(N) \).

Clearly, every graded strongly 2-absorbing second submodule is a graded 2-absorbing second submodule. This motivates the following question.

Question 3.2. Let \( M \) be a graded \( R \)-module. Is every graded 2-absorbing second \( R \)-submodule of \( M \) a graded strongly 2-absorbing second \( R \)-submodule of \( M \)?

Proposition 3.3. Let \( N \) be a graded \( R \)-submodule of \( M \). Then \( N \) is a graded strongly 2-absorbing second \( R \)-submodule of \( M \) if and only if for every \( x, y \in h(R) \), we have \( xyN = xN \) or \( xyN = yN \) or \( xyN = \{0\} \).

Proof. Suppose that \( N \) is a graded strongly 2-absorbing second \( R \)-submodule of \( M \). Then \( N = \{0\} \). Let \( x, y \in h(R) \). Then \( xyN \subseteq xyN \), which implies that \( xN \subseteq xyN \) or \( yN \subseteq xyN \) or \( xyN = \{0\} \). Thus \( xyN = xN \) or \( xyN = yN \) or \( xyN = \{0\} \). The converse is clear. □

Corollary 3.4. If \( N \) is a graded strongly 2-absorbing second \( R \)-submodule of \( M \), then \( \text{Ann}_R(N) \) is a graded 2-absorbing ideal of \( R \).

Proof. Let \( x, y, z \in h(R) \) such that \( xyz \in \text{Ann}_R(N) \). Then by Proposition 3.3, we have \( xyN = xN \) or \( xyN = yN \) or \( xyN = \{0\} \). If \( xyN = \{0\} \), then we are done. Suppose that \( xyN = xN \). Then \( zN \subseteq zxyN = \{0\} \). Similarly, if \( xyN = yN \). □

The following example shows that the converse of Corollary 3.4 is not true in general.
Example 3.5. In Example 2.3, \( N = \langle p \rangle \) (where \( p \) is a prime number) is a graded \( R \)-submodule of \( M = \mathbb{Z}[i] \) such that \( \text{Ann}_R(N) = \{0\} \) is a graded 2-absorbing ideal of \( R \), but \( N \) is not a graded strongly 2-absorbing second \( R \)-submodule of \( \mathbb{Z}[i] \).

Corollary 3.6. Let \( N \) be a graded strongly 2-absorbing second \( R \)-submodule of \( M \). If \( L \) is a graded \( R \)-submodule of \( M \) such that \( N \subseteq L \), then \( (L :_R N) \) is a graded 2-absorbing ideal of \( R \).

Proof. Let \( x, y, z \in h(R) \) such that \( xyz \in (L :_R N) \). Then \( xyzN \subseteq L \), and then \( xzN \subseteq L \) or \( yzN \subseteq L \) or \( xyzN = \{0\} \). If \( xzN \subseteq L \) or \( yzN \subseteq L \), then we are done. If \( xyzN = \{0\} \), then the result follows by Corollary 3.4. \( \square \)

A graded \( R \)-module \( M \) is said to be graded comultiplication if for every graded \( R \)-submodule \( N \) of \( M \) there exists a graded ideal \( J \) of \( R \) such that \( N = (0 :_M J) \), equivalently, for every graded \( R \)-submodule \( N \) of \( M \), we have \( N = (0 :_M \text{Ann}_R(N)) \). The concept of graded comultiplication modules was introduced by H. Ansari-Toroghy and F. Farshadifar in [10]. Some generalizations on graded comultiplication modules have been introduced in [3]. The next proposition shows that the converse of Corollary 3.4 is true if \( M \) is a graded comultiplication \( R \)-module.

Proposition 3.7. Let \( M \) be a graded comultiplication \( R \)-module. If \( L \) is a graded \( R \)-submodule of \( M \) such that \( \text{Ann}_R(L) \) is a graded 2-absorbing ideal of \( R \), then \( L \) is a graded strongly 2-absorbing second \( R \)-submodule of \( M \). In particular, \( L \) is a graded 2-absorbing second \( R \)-submodule of \( M \).

Proof. Let \( x, y \in h(R) \) and \( K \) be a graded \( R \)-submodule of \( M \) such that \( xyL \subseteq K \). Then \( \text{Ann}_R(K)xyL = \{0\} \). So, \( \text{Ann}_R(K)xL = \{0\} \) or \( \text{Ann}_R(K)yL = \{0\} \) or \( xyL = \{0\} \). If \( xyL = \{0\} \), then we are done. If \( \text{Ann}_R(K)xL = \{0\} \) or \( \text{Ann}_R(K)yL = \{0\} \), then \( \text{Ann}_R(K) \subseteq \text{Ann}_R(xL) \) or \( \text{Ann}_R(K) \subseteq \text{Ann}_R(yL) \). Hence, \( xL \subseteq K \) or \( yL \subseteq K \) since \( M \) is graded comultiplication. \( \square \)

Remark 3.8. Example 3.5 Shows that the condition that \( M \) is a graded comultiplication \( R \)-module in Proposition 3.7 is necessary since \( M \) is not a graded comultiplication \( R \)-module.

Corollary 3.9. Let \( M \) be a graded comultiplication \( R \)-module. If \( M \) is graded cocyclic, then every graded 2-absorbing second \( R \)-submodule of \( M \) is graded strongly 2-absorbing second.
Proof. Let $L$ be a graded 2-absorbing second $R$-submodule of $M$. Then by Corollary 2.13, $\text{Ann}_R(L)$ is a graded 2-absorbing ideal of $R$, and then the result follows by Proposition 3.7. □

Proposition 3.10. Let $N$ be a graded strongly 2-absorbing second $R$-submodule of $M$. If $(K \cap L :_R N)$ is a graded prime ideal of $R$ for all graded completely irreducible $R$-submodules $K$ and $L$ of $M$ with $N \subseteq K \cap L$, then $\text{Ann}_R(N)$ is a graded prime ideal of $R$.

Proof. Suppose that $\text{Ann}_R(N)$ is not a graded prime ideal of $R$. Then there exist $x, y \in h(R)$ such that $xyN = \{0\}$, $xN = \{0\}$ and $yN = \{0\}$. So, there exist graded completely irreducible $R$-submodules $K$ and $L$ of $M$ such that $xN \subseteq K$ and $yN \subseteq L$. Now, $xyN \subseteq K \cap L$, which implies that $xy \in (K \cap L :_R N)$, and then $xN \subseteq K \cap L$ or $yN \subseteq K \cap L$. In both cases, we have a contradiction. □

Lemma 3.11. Suppose that $f : M \rightarrow S$ is a graded $R$-homomorphism of graded $R$-modules.

1. If $f$ is a graded $R$-monomorphism and $K$ is a graded $R$-submodule of $f(M)$, then $f^{-1}(K)$ is a graded $R$-submodule of $M$.

2. If $L$ is a graded $R$-submodule of $M$ with $\text{Ker}(f) \subseteq L$, then $f(L)$ is a graded $R$-submodule of $f(M)$.

Proof.

1. Clearly, $f^{-1}(K)$ is an $R$-submodule of $M$. Let $x \in f^{-1}(K)$. Then $x \in M$ with $f(x) \in K$, and then $x = \sum_{g \in G} x_g$ where $x_g \in M_g$ for all $g \in G$. So, for every $g \in G$, $f(x_g) \in f(M_g) \subseteq S_g$ such that
\[
\sum_{g \in G} f(x_g) = f\left(\sum_{g \in G} x_g\right) = f(x) \in K.
\]
Since $K$ is graded, $f(x_g) \in K$ for all $g \in G$, i.e., $x_g \in f^{-1}(K)$ for all $g \in G$. Hence, $f^{-1}(K)$ is a graded $R$-submodule of $M$.

2. Clearly, $f(L)$ is an $R$-submodule of $f(M)$. Let $y \in f(L)$. Then $y \in f(M)$, and so there exists $x \in M$ such that $y = f(x)$, so $f(x) \in L$, which implies that $x \in L$ since $\text{Ker}(f) \subseteq L$, and hence $x_g \in L$ for all $g \in G$ since $L$ is graded. Thus, $y_g = (f(x))_g = f(x_g) \in f(L)$ for all $g \in G$. Therefore, $f(L)$ is a graded $R$-submodule of $S$. □
Lemma 3.12. Let \( f : M \rightarrow S \) be a graded monomorphism of graded \( R \)-modules. If \( N \) is a graded completely irreducible \( R \)-submodule of \( M \), then \( f(N) \) is a graded completely irreducible \( R \)-submodule of \( f(M) \).

**Proof.** Let \( \{L_i\}_{i \in \Delta} \) be a family of a graded \( R \)-submodules of \( f(M) \) such that \( f(N) = \bigcap_{i \in \Delta} L_i \). Then \( N = f^{-1}(f(N)) = f^{-1} \left( \bigcap_{i \in \Delta} L_i \right) = \bigcap_{i \in \Delta} f^{-1}(L_i) \).

So, there exists \( i \in \Delta \) such that \( N = f^{-1}(L_i) \) since \( N \) is a graded completely irreducible \( R \)-submodule of \( M \). Therefore, \( f(N) = f(f^{-1}(L_i)) = f(M) \cap L_i = L_i \), as needed. \( \square \)

Similarly, one can prove the next lemma.

Lemma 3.13. Let \( f : M \rightarrow S \) be a graded monomorphism of graded \( R \)-modules. If \( L \) is a graded completely irreducible \( R \)-submodule of \( f(M) \), then \( f^{-1}(L) \) is a graded completely irreducible \( R \)-submodule of \( M \).

Proposition 3.14. Let \( f : M \rightarrow S \) be a graded monomorphism of graded \( R \)-modules. If \( N \) is a graded strongly \( 2 \)-absorbing second \( R \)-submodule of \( M \) with \( \text{Ker}(f) \subseteq N \), then \( f(N) \) is a graded \( 2 \)-absorbing second \( R \)-submodule of \( f(M) \).

**Proof.** Since \( N = \{0\} \) and \( f \) is injective, we have \( f(N) = \{0\} \). Let \( x, y \in h(R) \) and \( K \) be a graded completely irreducible \( R \)-submodule of \( S \) such that \( xyf(N) \subseteq K \). Then \( xyN \subseteq f^{-1}(K) \), and then \( xN \subseteq f^{-1}(K) \) or \( yN \subseteq f^{-1}(K) \) or \( xyN = \{0\} \). Therefore, \( xf(N) \subseteq f(f^{-1}(K)) = f(M) \cap K \subseteq K \) or \( yf(N) \subseteq K \) or \( xyf(N) = \{0\} \), as needed. \( \square \)

Similarly, one can prove the next proposition.

Proposition 3.15. Let \( f : M \rightarrow S \) be a graded monomorphism of graded \( R \)-modules. If \( N \) is a graded \( 2 \)-absorbing second \( R \)-submodule of \( M \) with \( \text{Ker}(f) \subseteq N \), then \( f(N) \) is a graded \( 2 \)-absorbing second \( R \)-submodule of \( f(M) \).

Proposition 3.16. Let \( f : M \rightarrow S \) be a graded monomorphism of graded \( R \)-modules. If \( L \) is a graded strongly \( 2 \)-absorbing second \( R \)-submodule of \( f(M) \), then \( f^{-1}(L) \) is a graded \( 2 \)-absorbing second \( R \)-submodule of \( M \).
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Proof. If \( f^{-1}(L) = \{0\} \), then \( f(M) \cap L = f(f^{-1}(L)) = f(\{0\}) = \{0\} \). Thus \( L = \{0\} \) which is a contradiction. Therefore, \( f^{-1}(L) \neq \{0\} \). Let \( x, y \in h(R) \) and \( K \) be a graded completely irreducible \( R \)-submodule of \( M \) such that \( xyf^{-1}(L) \subseteq K \). Then \( xyL \subseteq xy(f(M) \cap L) = xyf(f^{-1}(L)) \subseteq f(K) \). So, \( xL \subseteq f(K) \) or \( yL \subseteq f(K) \) or \( xyL = \{0\} \). Hence, \( xf^{-1}(L) \subseteq K \) or \( yf^{-1}(L) \subseteq K \) or \( xyf^{-1}(L) = \{0\} \), as needed. □

Proposition 3.17. Let \( f : M \to S \) be a graded monomorphism of graded \( R \)-modules. If \( L \) is a graded 2-absorbing second \( R \)-submodule of \( f(M) \), then \( f^{-1}(L) \) is a graded 2-absorbing second \( R \)-submodule of \( M \).

4. Graded Quasi 2-Absorbing Second Submodules

In this section, we introduce and study the concept of graded quasi 2-absorbing second submodules.

Definition 4.1. Let \( M \) be a graded ring and \( N \) be a nonzero graded \( R \)-submodule of \( M \). Then \( N \) is said to be a graded quasi 2-absorbing second \( R \)-submodule of \( M \) if \( \text{Ann}_R(N) \) is a graded 2-absorbing ideal of \( R \).

Remark 4.2. By Corollary 3.4, every graded strongly 2-absorbing second \( R \)-submodule is a graded quasi 2-absorbing second \( R \)-submodule. But the converse is not true in general by Example 3.5.

Proposition 4.3. Let \( M \) be a graded comultiplication \( R \)-module. Then a graded \( R \)-submodule \( N \) of \( M \) is a graded strongly 2-absorbing second \( R \)-submodule of \( M \) if and only if it is a graded quasi 2-absorbing second \( R \)-submodule of \( M \).

Proof. It follows from Corollary 3.4 and Proposition 3.7. □

Proposition 4.4. Let \( M \) be a graded \( R \)-module and \( N \) be a graded quasi 2-absorbing second \( R \)-submodule of \( M \). If \( J \) is a graded ideal of \( R \) such that \( J \subseteq \text{Ann}_R(N) \), then \( JN \) is a graded quasi 2-absorbing second \( R \)-submodule of \( M \).
Proof. Since \( J \subseteq \text{Ann}_R(N) \), we have \( \text{Ann}_R(JN) \) is a proper graded ideal of \( R \). Let \( x, y, z \in h(R) \) such that \( xyz \in \text{Ann}_R(JN) \). Then \( xyzJN = \{0\} \), and then \( xzN = \{0\} \) or \( zyJN = \{0\} \) or \( xyJN = \{0\} \). If \( zyJN = \{0\} \) or \( xyJN = \{0\} \), then \( xzN = \{0\} \). Suppose that \( xzN = \{0\} \). Then \( xz \in \text{Ann}_R(N) \subseteq \text{Ann}_R(JN) \), so \( xzJN = \{0\} \), as required. \( \square \)

Proposition 4.5. Let \( M \) be a graded \( R \)-module and \( N \) be a graded quasi 2-absorbing second \( R \)-submodule of \( M \). Then \( \text{Ann}_R(J^nN) = \text{Ann}_R(J^{n+1}N) \) for all graded ideal \( J \) of \( R \) and for all \( n \geq 2 \).

Proof. Let \( J \) be a graded ideal of \( R \). It is enough to prove that \( \text{Ann}_R(J^2N) = \text{Ann}_R(J^3N) \). Clearly, \( \text{Ann}_R(J^2N) \subseteq \text{Ann}_R(J^3N) \). Since \( N \) is graded quasi 2-absorbing second, \( \text{Ann}_R(J^3N)J^2N = \{0\} \) implies that \( \text{Ann}_R(J^3N)J^2N = \{0\} \) or \( J^2N = \{0\} \). If \( \text{Ann}_R(J^3N)J^2N = \{0\} \), then \( \text{Ann}_R(J^3N) \subseteq \text{Ann}_R(J^2N) \). If \( J^2N = \{0\} \), then \( \text{Ann}_R(J^2N) = R = \text{Ann}_R(J^3N) \). \( \square \)

For a graded \( R \)-submodule \( N \) of \( M \), the graded second radical of \( N \) is defined as the sum of all graded second \( R \)-submodules of \( M \) contained in \( N \), and its denoted by \( G\text{Sec}^s(N) \). If \( N \) does not contain any graded second \( R \)-submodule, then \( G\text{Sec}^s(N) = \{0\} \). The set of all graded second \( R \)-submodules of \( M \) is called the graded second spectrum of \( M \), and is denoted by \( G\text{Spec}^s(M) \). On the other hand, the set of all graded prime \( R \)-submodules of \( M \) is called the graded spectrum of \( M \), and is denoted by \( G\text{Spec}(M) \). The map \( \phi : G\text{Spec}^s(M) \rightarrow G\text{Spec}(R/\text{Ann}_R(M)) \) defined by \( \phi(N) = \text{Ann}_R(N)/\text{Ann}_R(M) \) is called the natural map of \( G\text{Spec}^s(M) \), see \([9]\) and \([14]\).

For a graded ideal \( I \) of \( R \), the graded radical of \( I \) is defined to be the set of all \( r \in R \) such that for each \( g \in G \), there exists a positive integer \( n_g \) satisfies \( r^n \in I \), and it is denoted by \( \text{Grad}(I) \). One can see that if \( r \in h(R) \), then \( r \in \text{Grad}(I) \) if and only if \( r^n \in I \) for some positive integer \( n \), see \([22]\).

Lemma 4.6. Let \( N \) and \( K \) be two graded \( R \)-submodules of \( M \). Then

1. \( G\text{Sec}(N) \subseteq N \).
2. If \( N \subseteq K \), then \( G\text{Sec}(N) \subseteq G\text{Sec}(K) \).
3. \( G\text{Sec}((0 :_M I)) = G\text{Sec}((0 :_M Gr(I))) \) for all graded ideal \( I \) of \( R \).
4. \( G\text{Sec}(N) \subseteq (0 :_M Gr(\text{Ann}_R(N))) \).
Proof. (1) and (2) are straightforward. (3) Let $I$ be a graded ideal of $R$. If $GSec((0 :_M I)) = \{0\}$, then we are done. Suppose that there exists a graded second $R$-submodule $L$ of $M$ such that $L \subseteq (0 :_M I)$. Then $I \subseteq Ann_R(L)$. Since $Ann_R(L)$ is a graded prime ideal of $R$, we have $Gr(I) \subseteq Ann_R(L)$. Hence, $L \subseteq (0 :_M Ann_R(L)) \subseteq (0 :_M Gr(I))$. Consequently, $GSec((0 :_M I)) \subseteq GSec((0 :_M Gr(I))$. The reverse inclusion follows by part (2). (4) Since $N \subseteq (0 :_M Ann_R(N))$, the result follows by parts (1), (2) and (3). □

Lemma 4.7. Let $M$ be a graded $R$-module and $N$ be a graded $R$-submodule of $M$. If the natural map $\phi$ of $GSpec^s(N)$ is surjective, then $Ann_R(GSec(N)) = Gr(Ann_R(N))$.

Proof. If $N = \{0\}$, then we are done. Suppose that $N \neq \{0\}$. Then by Lemma 4.6 part (4), we have $Gr(Ann_R(N)) \subseteq Ann_R(GSec(N))$. Assume that $Gr(Ann_R(N)) = \bigcap_i P_i$ where $P_i$ is a graded prime ideal of $R$ containing $Ann_R(N)$. Since $\phi$ is surjective, for every $P_i$, there exists $L_i \in GSpec^s(N)$ such that $Ann_R(L_i) = P_i$. Hence, $\bigcap_i L_i \subseteq GSec(N)$. Therefore, $Ann_R(GSec(N)) \subseteq Ann_R\left(\bigcap_i L_i\right) = \bigcap_i P_i = Gr(Ann_R(N))$, as needed. □

Proposition 4.8. Let $N$ be a graded quasi 2-absorbing second $R$-submodule of $M$. If the natural map $\phi$ of $GSpec^s(N)$ is surjective, then $GSec(N)$ is a graded quasi 2-absorbing second $R$-submodule of $M$.

Proof. By Lemma 4.7, $Ann_R(GSec(N)) = Gr(Ann_R(N))$, and then the result follows from the fact that $Gr(Ann_R(N))$ is a graded 2-absorbing ideal of $R$ by ([6], Lemma 2.5). □

Proposition 4.9. Let $M$ be a graded comultiplication $R$-module, $N \subseteq L$ be two graded $R$-submodules of $M$ and $L$ be a graded quasi 2-absorbing second $R$-submodule of $M$. Then $L/N$ is a graded quasi 2-absorbing second $R$-submodule of $M/N$.

Proof. Let $x,y,z \in h(R)$ such that $xyz(L/N) = \{0\}$. Then $xyzL \subseteq N$, and then $Ann_R(N)xyzL = \{0\}$. Thus, $Ann_R(N)xyL = \{0\}$ or $Ann_R(N)xzL = \{0\}$ or $yzL = \{0\}$. If $yzL = \{0\}$, then $yz(L/N) = \{0\}$, and then we are
done. If \( \text{Ann}_R(N)xyL = \{0\} \) or \( \text{Ann}_R(N)xzL = \{0\} \), then \( xyL \subseteq (0 :_M \text{Ann}_R(N)) \) or \( xzL \subseteq (0 :_M \text{Ann}_R(N)) \). Since \( M \) is graded comultiplication, we have \( N = (0 :_M \text{Ann}_R(N)) \), and the result follows obviously. □

The next example shows that the condition \( M \) is a graded comultiplication \( R \)-module is necessary in Proposition 4.9.

**Example 4.10.** Let \( R = \mathbb{Z} \), \( M = \mathbb{Z}[i] \) and \( G = \mathbb{Z}_2 \). Then \( R \) is \( G \)-graded by \( R_0 = \mathbb{Z} \) and \( R_1 = \{0\} \). Also, \( M \) is \( G \)-graded by \( M_0 = \mathbb{Z} \) and \( M_1 = i\mathbb{Z} \).

Clearly, \( \mathbb{Z} \) is a graded quasi 2-absorbing second \( R \)-submodule of \( M \). On the other hand, \( 12\mathbb{Z} \) is a graded \( R \)-submodule of \( M \) such that \( 12\mathbb{Z} \subseteq \mathbb{Z} \) and \( \mathbb{Z}/12\mathbb{Z} \) is not graded quasi 2-absorbing second. Note that, \( M \) is not graded comultiplication \( R \)-module.

**Proposition 4.11.** Let \( f : M \to S \) be a graded monomorphism of graded \( R \)-modules. Then \( N \) is a graded quasi 2-absorbing second \( R \)-submodule of \( M \) if and only if \( f(N) \) is a graded quasi 2-absorbing second \( R \)-submodule of \( S \).

**Proof.** It follows from the fact that \( \text{Ann}_R(N) = \text{Ann}_R(f(N)) \). □

**References**


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