On $r$– dynamic coloring of the gear graph families

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Received: September 2019 | Accepted: April 2020

Abstract:

An $r$–dynamic coloring of a graph $G$ is a proper coloring $c$ of the vertices such that $|c(N(v))| \geq \min \{r, d(v)\}$, for each $v \in V(G)$. The $r$–dynamic chromatic number of a graph $G$ is the minimum $k$ such that $G$ has an $r$–dynamic coloring with $k$ colors. In this paper, we obtain the $r$–dynamic chromatic number of the middle, central and line graphs of the gear graph.

Keywords: $r$– dynamic coloring; gear graph; middle graph; central graph and line graph.

MSC (2020): 05C15.

Cite this article as (IEEE citation style):


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1. Introduction

Graphs in this paper are simple and finite. For undefined terminologies and notations see [5, 17]. Thus for a graph $G$, $\delta(G), \Delta(G)$ and $\chi(G)$ denote the minimum degree, maximum degree and chromatic number of $G$ respectively. When the context is clear we write, $\delta, \Delta$ and $\chi$ for brevity. For $v \in V(G)$, let $N(v)$ denote the set of vertices adjacent to $v$ in $G$ and $d(v) = |N(v)|$. The $r$-dynamic chromatic number was first introduced by Montgomery [14].

An $r$-dynamic coloring of a graph $G$ is a map $c$ from $V(G)$ to the set of colors such that (i) if $uv \in E(G)$, then $c(u) \neq c(v)$ and (ii) for each vertex $v \in V(G)$, $|c(N(v))| \geq \min \{r, d(v)\}$, where $N(v)$ denotes the set of vertices adjacent to $v$ and $d(v)$ its degree and $r$ is a positive integer.

The first condition characterizes proper colorings, the adjacency condition and second condition is double-adjacency condition. The $r$-dynamic chromatic number of a graph $G$, written $\chi_r(G)$, is the minimum $k$ such that $G$ has an $r$-dynamic proper $k$-coloring. The 1-dynamic chromatic number of a graph $G$ is equal to its chromatic number. The 2-dynamic chromatic number of a graph has been studied under the name dynamic chromatic number denoted by $\chi_d(G)$ [1, 2, 3, 4, 8]. By simple observation, we can show that $\chi_r(G) \leq \chi_{r+1}(G)$, however $\chi_{r+1}(G) - \chi_r(G)$ can be arbitrarily large, for example $\chi(Petersen) = 2$, $\chi_d(Petersen) = 3$, but $\chi_3(Petersen) = 10$. Thus, finding an exact values of $\chi_r(G)$ is not trivially easy.

There are many upper bounds and lower bounds for $\chi_d(G)$ in terms of graph parameters. For example, for a graph $G$ with $\Delta(G) \geq 3$, Lai et al.[8] proved that $\chi_d(G) \leq \Delta(G)+1$. An upper bound for the dynamic chromatic number of a $d$-regular graph $G$ in terms of $\chi(G)$ and the independence number of $G$, $\alpha(G)$, was introduced in [7]. In fact, it was proved that $\chi_d(G) \leq \chi(G) + 2\log_2\alpha(G) + 3$. Taherkhani gave in [15] an upper bound for $\chi_2(G)$ in terms of the chromatic number, the maximum degree $\Delta$ and the minimum degree $\delta$. i.e., $\chi_2(G) - \chi(G) \leq \left\lfloor (\Delta e)/\delta \log \left(2e\left(\Delta^2 + 1\right)\right)\right\rfloor$.

Li et al.proved in [10] that the computational complexity of $\chi_d(G)$ for a 3-regular graph is an NP-complete problem. Furthermore, Li and Zhou [9] showed that to determine whether there exists a 3-dynamic coloring, for a claw free graph with the maximum degree 3, is NP-complete.

N.Mohanapriya et al. [11, 12] studied the dynamic chromatic number for various graph families. Also, it was proven in [13] that the $r$-dynamic chromatic number of line graph of a helm graph $H_n$.

In this paper, we study $\chi_r(G)$, when $1 \leq r \leq \Delta$. We find the $r$- dynamic chromatic number of the middle, central and line graphs of the gear graph.
2. Preliminaries

Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The middle graph $M(G)$ of $G$, denoted by $M(G)$, is defined as follows. The vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices $x, y$ of $M(G)$ are adjacent in $M(G)$ in one of the following cases: (i) $x, y$ are in $E(G)$ and $x, y$ are adjacent in $G$. (ii) $x$ is in $V(G)$, $y$ is in $E(G)$, and $x, y$ are incident in $G$.

The central graph $C(G)$ of a graph $G$ is obtained from $G$ by adding an extra vertex on each edge of $G$, and then joining each pair of vertices of the original graph which were previously non-adjacent.

The line graph $L(G)$ of $G$ denoted by $L(G)$ is the graph with vertices are the edges of $G$ with two vertices of $L(G)$ adjacent whenever the corresponding edges of $G$ are adjacent.

The gear graph is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle. The gear graph $G_n$ has $2n + 1$ nodes and $3n$ edges.

Let $V(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(G_n) = \{vv_i : 1 \leq i \leq n\} \cup \{u_iv_i : 1 \leq i \leq n\} \cup \{u_iu_{i+1} : 1 \leq i \leq n\}$ and the meaning of mod $n$ is the obvious.

3. Main Theorem

**Theorem 3.1.** Let $n \geq 5$, $M(G_n)$ be the middle graph of a gear graph $G_n$ and let $\Delta = \Delta(M(G_n))$. Then

$$
\chi_r(M(G_n)) = \begin{cases}
 n + 1, & 1 \leq r \leq 4 \\
 n + 2, & 5 \leq r \leq \Delta - 2 \\
 n + 4, & r = \Delta - 1 \text{ and } n \equiv 0 \mod 3 \\
 n + 5, & r = \Delta - 1 \text{ and } n \equiv 1 \mod 3 \\
 n + 4, & r = \Delta - 1 \text{ and } n \equiv 2 \mod 3 \\
 n + 5, & r = \Delta \text{ and } n \equiv 0 \mod 3 \\
 n + 7, & r = \Delta \text{ and } n \equiv 1 \mod 3 \\
 n + 6, & r = \Delta \text{ and } n \equiv 2 \mod 3
\end{cases}
$$

**Proof.** By the definition of middle graph, $V(M(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq 2n\}$.

The vertices $v$ and $\{e_i : 1 \leq i \leq n\}$ induces a clique of order $K_{n+1}$ in $M(G_n)$.
Thus, $\chi_\delta(M(G_n)) \geq n + 1$.

We divide the proof into some cases.

**Case 1:** For $1 \leq r \leq 4$

The $r$-dynamic $(n + 1)$ coloring is as follows:

For $1 \leq i \leq n$, assign the color $c_i$ to $e_i$ and assign the color $c_{n+1}$ to $v_i$.

For $1 \leq i \leq n$, assign the color $c_{n+1}$ to $u_i$ and $v_i$.

- $|N(u_i)| = d(u_i) = 2 = \delta$,
- $|N(v_i)| = d(v_i) = 3$
- $|N(u)| = d(u) = n$,
- $|N(e_i)| = d(e_i) = n + 3$
- $|N(s_i)| = d(s_i) = 5$

For $1 \leq i \leq 2n$, assign the allowed colors to the vertex $s_i$ and also it must satisfies the $r-$ adjacency condition.

- color the vertices $s_1, s_3, s_5, s_7, \cdots s_{2n-5}, s_{2n-3}, s_{2n-1}$ with colors $c_3, c_4, c_5, \cdots c_n, c_1, c_2$ (the order of assigned color is important).

- color the vertices $s_2, s_4, s_6, s_8, \cdots s_{2n-4}, s_{2n-2}, s_{2n}$ with colors $c_n, c_1, c_2, c_3, \cdots c_{n-3}, c_{n-2}, c_{n-1}$ (the order of assigned color is important).

We know that $|N(v)| = d(v) = n$, so we need the color $n + 1$.

It is easy to verify that adjacency and $r$-adjacency conditions are fulfilled.

Hence, $\chi_r(M(G_n)) = n + 1$, for $n \geq 5$ and $1 \leq r \leq 4$.

**Case 2:** For $5 \leq r \leq \Delta - 2$

The $r$-dynamic $(n + 2)$ coloring is as follows:

For $1 \leq i \leq n$, assign the color $c_i$ to $e_i$ and assign the color $c_{n+1}$ to $v_i$.

For $1 \leq i \leq n$, assign the color $c_{n+1}$ to $u_i$.

For $1 \leq i \leq 2n$, if any, assign the vertex $s_i$ to one of the allowed colors - such color exists, because $|N(s_i)| = d(s_i) = 5$

- color the vertices $s_1, s_3, s_5, s_7, \cdots s_{2n-5}, s_{2n-3}, s_{2n-1}$ with colors $c_3, c_4, c_5, \cdots c_n, c_1, c_2$ (the order of assigned color is important).

- color the vertices $s_2, s_4, s_6, s_8, \cdots s_{2n-4}, s_{2n-2}, s_{2n}$ with colors $c_n, c_1, c_2, c_3, \cdots c_{n-3}, c_{n-2}, c_{n-1}$ (the order of assigned color is important).
• color the vertex \( v_i \) with the color \( c_{n+2} \).

Now \(|N(s_i)|\) satisfies the \( r \)-adjacency condition.
But \( d(e_i) = n + 3 \), so \( N(e_i) \) having \( n + 2 \) colors.
It is easy to verify that the \( r \)-adjacency condition is fulfilled.
Hence, \( \chi_r(M(G_n)) = n + 2 \), for \( n \geq 5 \) and \( 5 \leq r \leq \Delta - 2 \).

**Case 3 :** For \( r = \Delta - 1 \) and \( n \equiv 0 \mod 3 \)

The \( r \)-dynamic \((n + 4)\) coloring is as follows:
For \( 1 \leq i \leq n \), assign the color \( c_i \) to \( e_i \) and assign the color \( c_{n+1} \) to \( v \).
For \( 1 \leq i \leq n \), assign the color \( c_{n+1} \) to \( u_i \).
For \( 1 \leq i \leq n \), assign the color \( c_{n+2} \) to \( v_i \).
\(|N(e_i)|\) having \( n + 1 \) colors only. So we assign one new color to \( s_i \).

• color the vertices \( s_1, s_4, s_7, s_{10}, \ldots s_{2n-5}, s_{2n-2} \) with color \( c_{n+3} \).

• color the vertices \( s_2, s_5, s_8, s_{11}, \ldots s_{2n-4}, s_{2n-1} \) with colors \( c_{n+4} \).

Now \( s_3, s_6, s_9, \ldots s_{2n-3}, s_{2n} \) are uncolored. So assign these vertices to any one of the allowed colors-such color exists.

• color the vertices \( s_3, s_6, s_9, \ldots s_{2n-3}, s_{2n} \) with colors \( c_5, c_7, c_9, \ldots c_{n-1}, c_1, c_3 \) (the order of assigned color is important).

Now neighbours of \( e_i \) having \( n + 4 \) colors and an easy check shows that the \( r \)-adjacency condition is fulfilled.

Hence, \( \chi_r(M(G_n)) = n + 4 \), for \( n \geq 5 \), \( r = \Delta - 1 \) and \( n \equiv 0 \mod 3 \).

**Case 4 :** For \( r = \Delta - 1 \) and \( n \equiv 1 \mod 3 \)

The \( r \)-dynamic \((n + 5)\) coloring is as follows:
For \( 1 \leq i \leq n \), assign the color \( c_i \) to \( e_i \) and assign the color \( c_{n+1} \) to \( v \).
For \( 1 \leq i \leq n \), assign the color \( c_{n+1} \) to \( u_i \).
For \( 1 \leq i \leq n \), assign the color \( c_{n+2} \) to \( v_i \).
\( N(e_i) \) having \( n + 1 \) colors. So we have to assign one new color to \( s_i \).

• color the vertices \( s_1, s_4, s_7, s_{10}, \ldots s_{2n-4} \) with color \( c_{n+3} \).

• color the vertices \( s_2, s_5, s_8, s_{11}, \ldots s_{2n-3} \) with color \( c_{n+4} \).
But neighbours of $e_n$ having $n + 1$ colors only. So we have to assign a new color $c_{n+5}$ to $s_{2n-2}$.

Now neighbours of $e_i$ having $n + 2$ colors. But the vertices $s_{2n-1}$ and $s_{2n}$ are uncolored.

So we have to assign any one of the allowed colors to $s_{2n-1}$ and $s_{2n}$.

- color the vertex $s_{2n-1}$ with the color $c_2$ and color the vertex $s_{2n}$ with the color $c_3$.

Now an easy check shows that the $r$-adjacency condition is fulfilled.

Hence, $\chi_r(M(G_n)) = n + 5$, for $n \geq 5$ and $r = \Delta - 1$ and $n \equiv 1 \mod 3$.

**Case 5 : For $r = \Delta - 1$ and $n \equiv 2 \mod 3$**

The $r$- dynamic $(n + 4)$ coloring is as follows:

- For $1 \leq i \leq n$, assign the color $c_i$ to $e_i$ and assign the color $c_{n+1}$ to $v$.
- For $1 \leq i \leq n$, assign the color $c_{n+1}$ to $u_i$.
- For $1 \leq i \leq n$, assign the color $c_{n+2}$ to $v_i$.

Now $N(e_i)$ having $n + 1$ colors.so we have to assign one new color to $s_i$.

- color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-3}$ with color $c_{n+3}$.
- color the vertices $s_2, s_5, s_8, s_11, \cdots s_{2n-2}$ with color $c_{n+4}$

But the vertices $s_3, s_6, s_9, \cdots s_{2n-4}, s_{2n-1}$ and $s_{2n}$ are uncolored. So we have to assign any one of the allowed colors to these vertices.

- color the vertices $s_3, s_6, s_9, \cdots s_{2n-1}, s_{2n}$ with colors $c_4, c_1, c_7, c_3, \cdots, c_2, c_8$ respectively.(the order of assigned color is important).

Now an easy check shows that the $r$- adjacency condition is fulfilled.

Hence, $\chi_r(M(G_n)) = n + 4$, for $n \geq 5$, $r = \Delta - 1$ and $n \equiv 2 \mod 3$.

**Case 6 : For $r = \Delta$ and $n \equiv 0 \mod 3$**

The $r$- dynamic $(n + 5)$ coloring is as follows:

- For $1 \leq i \leq n$, assign the color $c_i$ to $e_i$ and assign the color $c_{n+1}$ to $v$.
- For $1 \leq i \leq n$, assign the color $c_{n+1}$ to $u_i$.
- For $1 \leq i \leq n$, assign the color $c_{n+2}$ to $v_i$.
- For $r = \Delta$, we have to assign two new colors to neighbours of $e_i$. 
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- color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-5}, s_{2n-2}$ with color $c_{n+3}$
- color the vertices $s_2n, s_3, s_6, s_9, \cdots s_{2n-3}$ with color $c_{n+4}$
- color the vertices $s_2, s_5, s_8, s_{11}, \cdots s_{2n-1}$ with color $c_{n+5}$

Now an easy check shows that the $r-$adjacency condition is fulfilled for all the vertices.
Hence, $\chi_r(M(G_n)) = n + 5$, for $n \geq 5$, $r = \Delta$ and $n \equiv 0 \mod 3$.

**Case 7 :** For $r = \Delta$ and $n \equiv 1 \mod 3$

The $r-$ dynamic $(n + 7)$ coloring is as follows:
- For $1 \leq i \leq n$, assign the color $c_i$ to $e_i$ and assign the color $c_{n+1}$ to $v_i$.
- For $1 \leq i \leq n$, assign the color $c_{n+1}$ to $v_i$.
- For $1 \leq i \leq n$, assign the color $c_{n+2}$ to $v_i$.
- For $r = \Delta$, we have to assign two new colors to neighbours of $e_i$.
  - color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-4}$ with color $c_{n+3}$.
  - color the vertices $s_2n, s_3, s_6, s_9, \cdots s_{2n-5}$ with color $c_{n+4}$.
  - color the vertices $s_2, s_5, s_8, s_{11}, \cdots s_{2n-3}$ with color $c_{n+5}$.

But neighbours of $e_n$ does not satisfies the $r$-adjacency condition.
So we have to assign two new colors to the vertices $s_{2n-2}$ and $s_{2n-1}$ respectively.
  - color the vertex $s_{2n-2}$ with the color $c_{n+6}$ and color the vertex $s_{2n-1}$ with the color $c_{n+7}$.

So we have to assign any one of the allowed colors to $s_{2n-1}$ and $s_{2n}$.
Now an easy check shows that the $r$-adjacency condition is fulfilled.
Hence, $\chi_r(M(G_n)) = n + 7$, for $n \geq 5$, $r = \Delta$ and $n \equiv 1 \mod 3$.

**Case 8 :** For $r = \Delta$ and $n \equiv 2 \mod 3$

The $r-$ dynamic $(n + 6)$ coloring is as follows:
- For $1 \leq i \leq n$, assign the color $c_i$ to $e_i$ and assign the color $c_{n+1}$ to $v_i$.
- For $1 \leq i \leq n$, assign the color $c_{n+1}$ to $u_i$.
- For $1 \leq i \leq n$, assign the color $c_{n+2}$ to $v_i$.
- For $r = \Delta$, we have to assign two new colors to neighbours of $e_i$. 
• color the vertices \( s_1, s_4, s_7, s_{10}, \ldots, s_{2n-3} \) with color \( c_{n+3} \).

• color the vertices \( s_{2n}, s_3, s_6, s_9, \ldots, s_{2n-4} \) with color \( c_{n+4} \).

• color the vertices \( s_2, s_5, s_8, s_{11}, \ldots, s_{2n-2} \) with color \( c_{n+5} \).

Now neighbours of \( e_n \) does not satisfies the \( r \)- adjacency condition.

• color the vertex \( s_{2n-1} \) with the new color \( c_{n+6} \).

Now an easy check shows that the \( r \)- adjacency condition is fulfilled. Hence, \( \chi_r(M(G_n)) = n + 6 \), for \( n \geq 5 \), \( r = \Delta \) and \( n \equiv 2 \mod\ 3 \). \( \square 

\textbf{Theorem 3.2.} Let \( n \geq 5 \), \( C(G_n) \) be the central graph of a Gear graph \( G_n \) and let 
\( \Delta = \Delta(C(G_n)) \). Then

\[
\chi_r(C(G_n)) = \begin{cases} 
  n + 1, & r = 1 \\
  2n + 1, & 0 \leq r \leq \Delta - 2 \\
  2n + 2, & r = \Delta - 1 \\
  3n + 3, & r = \Delta 
\end{cases}
\]

\textbf{Proof.} By the definition of central graph, subdividing each edge of \( G_n \) exactly once and then joining each pair of vertices of \( G_n \) which were non-adjacent.

Let \( V(C(G_n)) = V(G_n) \cup E(G_n) = \{v\} \cup \{v_i : 1 \leq i \leq n\} \cup \{u_i : 1 \leq i \leq n\} \cup \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq 2n\} \)

We divide the proof into some cases.

\textbf{Case 1 :} For \( r = 1 \)

The \( r \)- dynamic \( (n + 1) \)- coloring is as follows:

For \( 1 \leq i \leq n \), assign the color \( c_i \) to \( v_i \) and \( u_i \).

For \( 1 \leq i \leq n - 1 \), assign the color \( c_{i+1} \) to \( e_i+1 \) and assign the color \( c_n \) to \( e_1 \).

\[
|N(u_i)| = d(u_i) = 2n \\
|N(v_i)| = d(v_i) = 2n \\
|N(v)| = d(v) = 2n \\
|N(e_i)| = d(e_i) = 2 \\
\text{and } |N(s_i)| = d(s_i) = 2
\]

For \( 1 \leq i \leq 2n \), assign the color \( c_{n+1} \) to the vertex \( s_i \) and assign the color \( c_{n+1} \) to \( v \).
Now an easy check shows that the $r-$ adjacency condition is fulfilled. Hence, $\chi_r(C(G_n)) = n + 1$, for $r = 1$.

**Case 2**: For $\delta \leq r \leq \Delta - 2$

The $r-$ dynamic $(2n + 1)-$ coloring is as follows:

For $1 \leq i \leq n$, assign the color $c_i$ to $v_i$.

For $1 \leq i \leq 2n$, assign the color $c_{n+1}$ to $s_i$.

For $1 \leq i \leq n - 1$, assign the color $c_i$ to $e_{i+1}$ and assign the color $c_n$ to $e_1$ and also assign the color $c_{n+1}$ to $v$.

- Color the vertices $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$ with colors $c_{n+2}, c_{n+3}, \ldots, c_{2n}, c_{2n+1}$ (the order of assigned color is important).

Now an easy check shows that the $r-$adjacency condition is fulfilled. Hence, $\chi_r(C(G_n)) = 2n + 1$, for $\delta \leq r \leq \Delta - 2$.

**Case 3**: For $r = \Delta - 1$

The $r-$ dynamic $(2n + 2)-$ coloring is as follows:

For $1 \leq i \leq n$, assign the color $c_i$ to $v_i$ and assign the color $c_{n+1}$ to $v$.

For $1 \leq i \leq n - 1$, assign the color $c_i$ to $e_{i+1}$ and assign the color $c_n$ to $e_1$.

- Color the vertices $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$ with colors $c_{n+2}, c_{n+3}, \ldots, c_{2n}, c_{2n+1}$ (the order of assigned color is important).

- Color the vertices $s_2, s_4, s_6, \ldots, s_{2n-2}, s_{2n}$ with color $c_{n+1}$.

- Color the vertices $s_1, s_3, s_5, \ldots, s_{2n-3}, s_{2n-1}$ with colors $c_{2n+2}$.

Now an easy check shows that the $r-$adjacency condition is fulfilled. Hence, $\chi_r(C(G_n)) = 2n + 2$, for $r = \Delta - 1$.

**Case 4**: For $r = \Delta$

The $r-$ dynamic $(3n + 3)-$ coloring is as follows:

For $1 \leq i \leq n$, assign the color $c_i$ to $v_i$ and assign the color $c_{n+1}$ to $v$.

- Color the vertices $u_1, u_2, u_3, \ldots, u_{n-1}, u_n$ with colors $c_{n+2}, c_{n+3}, \ldots, c_{2n}, c_{2n+1}$ (the order of assigned color is important).
• Color the vertices \(s_1, s_3, s_5, \ldots s_{2n-3}, s_{2n-1}\) with colors \(c_{2n+2}\) and color the vertices \(s_2, s_4, s_6, \ldots s_{2n-2}, s_{2n}\) with color \(c_{2n+3}\).

• Color the vertices \(e_1, e_2, e_3, \ldots e_{n-1}, e_n\) with colors \(c_{2n+4}, c_{2n+5}, c_{2n+6} \ldots c_{3n+2}, c_{3n+3}\) respectively.(the order of assigned color is important).

Now an easy check shows that the \(r\)-adjacency condition is fulfilled. Hence, \(\chi_r(C(G_n)) = 3n + 3\), for \(r = \Delta\) \(\square\)

**Result:**
Let us consider the line graphs built on the base of Gear graph.

By the definition of line graph

\[V(L(G_n)) = E(G_n) = \{e_i : 1 \leq i \leq n\} \cup \{s_i : 1 \leq i \leq 2n\} \]  

Note that \(d(e_i) = n + 1, d(s_i) = 3\). Hence \(\delta(L(G_n)) = 3\).

Next, observe that the vertices \(\{e_1, e_2, e_3, \ldots, e_n\}\) induces a clique \(K_n\) in \(L(G_n)\). Thus,

\[\chi_\delta(L(G_n)) \geq n\]  

for any \(r\). Let us start with \(r = \delta\).

**Proposition 3.3.** Let \(n \geq 5\). Let \(L(G_n)\) be the line graph of a Gear graph \(G_n\). Then \(\chi_\delta(L(G_n)) = n\).

**Proof.** Due to (1), we have \(\chi_\delta(L(G_n)) \geq n\).

So, we need to fix only appropriate coloring.

For \(1 \leq i \leq n\), assign the color \(i\) to \(e_i\). Next, assign the colors to \(s_i\) such that partial coloring is proper and the \(r\)-adjacency condition for \(r = \delta\) is also fulfilled.

That is we should assign one of the allowed colors from \(\{1, 2, \ldots n\}\) to vertex \(s_i\) of degree 3, \(1 \leq i \leq n\).

The coloring we obtained is \(\delta\)– dynamic coloring of \(L(G_n)\).

The result from proposition can be extended to \(r\)-dynamic coloring for line graph of Gear graph for all \(r\), where \(1 \leq r \leq \Delta\). \(\square\)

**Theorem 3.4.** Let \(n \geq 6\), \(L(G_n)\) be the line graph of a Gear graph \(G_n\) and let \(\Delta = \Delta(L(G_n))\). Then
\[
\chi_r(L(G_n)) = \begin{cases}
  n, & 1 \leq r \leq n - 1 \\
n + 2, & r = n \text{ and } n \not\equiv 1 \mod 3 \\
n + 3, & r = n \text{ and } n \equiv 1 \mod 3 \\
n + 3, & r = n + 1 = \Delta, \ n \geq 5 \text{ and } 2n \equiv 0 \mod 3 \\
n + 4, & r = n + 1 = \Delta, \ n \geq 5 \text{ and } 2n \equiv 1 \mod 3 \\
n + 5, & r = n + 1 = \Delta, \ n \geq 5 \text{ and } 2n \equiv 2 \mod 3
\end{cases}
\]

**Proof.** We divide the proof into some cases.

**Case 1:** For \(1 \leq r \leq n - 1\)

The \(r\)-dynamic \((n)\)-coloring is as follows:

\[
|N(e_i)| = d(e_i) = n - 1, \\
|N(s_i)| = d(s_i) = 3 = \delta.
\]

Now an easy check shows that the \(r\)-adjacency condition is fulfilled.

Hence, \(\chi_r(L(G_n)) = n\), for \(1 \leq r \leq n - 1\)

**Case 2:** For \(r = n\) and \(n \not\equiv 1 \mod 3\)

The \(r\)-dynamic \((n + 2)\)-coloring is as follows:

- Color vertex \(e_i\) with color \(i\), \(1 \leq i \leq n\).

Let us notice that vertices adjacent to each vertex \(e_i\) must be colored with \(r = n\) different colors. After this step each vertex \(e_i\) has \(n - 1\) neighbours in different colors and exactly its two neighbours are uncolored: \(s_{i-1}, s_i\).

We have to color them with at least one new color to vertex \(s_i\) to fulfill \(r\)-adjacent condition for vertex \(s_i\). So \(\chi_r(L(G_n)) \geq n + 2\).

To color vertices \(s_i, 1 \leq i \leq n\).

Now the number of vertices \(s_i\), forming a cycle \(C_{2n}\), is not divisible by 3, so color the vertices \(s_1, s_4, s_7, s_{10}, \ldots, s_{2n-2}\) with color \(n + 1\).

Now another neighbour of \(e_1\) has uncolored. So we have to assign one of the allowed colors \(c_1, c_2, c_3, \ldots, c_n\) to vertex \(s_{2n}\).

Next, the two neighbours of \(e_2\) are uncolored. We have to color them with at least one new color to vertex \(s_2\) to fulfill \(r\)-adjacent condition for vertex \(e_i\).

- color the vertices \(s_2, s_5, s_8, \ldots, s_{2n-1}\) with color \(n + 2\).
Now the neighbours of $e_i$ has at least $n$ colors.
Now $s_3, s_6, s_9, s_{12}, \cdots s_{2n}$ vertices get any one of the allowed colors $c_1, c_2, c_3, \cdots c_n$.
Now an easy check shows that the $r$-adjacency condition is fulfilled.
Hence, $\chi_r(L(G_n)) = n + 2$, for $r = n$ and $n \not\equiv 1 \mod 3$

**Case 3 :** For $r = n$ and $n \equiv 1 \mod 3$
The $r$-dynamic $(n + 3)$-coloring is as follows:

- Color vertex $e_i$ with color $i$, $1 \leq i \leq n$.

Let us notice that vertices adjacent to each vertex $e_i$ must be colored with $r = n$ different colors. After this step each vertex $e_i$ has $n - 1$ neighbours in different colors and exactly its two neighbours are uncolored: $s_{i-1}, s_i$.

We have to color them with at least one new color to vertex $s_i$ to fulfill $r$-adjacent condition for vertex $s_i$, so $\chi_r(L(G_n)) \geq n + 2$.

To color vertices $s_i$, $1 \leq i \leq n$.

Now the number of vertices $s_i$, forming a cycle $C_{2n}$, is not divisible by 3, so color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-4}$ with color $n + 1$.

Now another neighbour of $e_1$ has uncolored. So we have to assign one of the allowed colors $1, 2, 3, \cdots n$ to vertex $s_{2n}$.

Next, the two neighbours of $e_2$ are uncolored. We have to color them with at least one new color to vertex $s_2$ to fulfill $r$-adjacent condition for vertex $e_i$.

- Color the vertices $s_2, s_5, s_8, \cdots s_{2n-3}$ with color $n + 2$.

But the neighbours of $e_n$ having only $n - 1$ colors. So we have to assign any one of the new color to the vertices $s_{2n-1}, s_{2n-2}$.

Suppose to assign color $n + 3$ to $s_{2n-2}$, next assign the uncolored vertices to the any one of the allowed colors $1, 2, \cdots n$ to fulfill $r$-adjacent condition for vertex $e_i$.

Now an easy check shows that the $r$-adjacency condition is fulfilled.
Hence, $\chi_r(L(G_n)) = n + 3$, for $r = n$ and $n \equiv 1 \mod 3$

**Case 4 :** $r = n + 1 = \Delta$ and $2n \equiv 0 \mod 3$
The $r$ dynamic $(n + 3)$-coloring is as follows:

- Color the vertex $e_i$ with color $i$, $1 \leq i \leq n$. 
It is clear that to color $2n$ remaining vertices: $s_i$ we have to use colors $n, \cdots \chi_r$.
we have to still take care of the $r$-adjacency condition for all vertices.

The $r$-adjacency condition for vertices $s_i, 1 \leq i \leq n$, we must use atleast two new colors to vertex $s_i$. So $\chi_r(L(G_n)) \geq n + 3$.

- Color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-2}$ with color $n+1$.
- Color the vertices $s_3, s_6, s_9, s_{12}, \cdots s_{2n}$ with new color $n+2$.

Now the vertex $s_2$ is uncolored. So we have to assign the new color $n + 3$ to the vertices $s_2, s_5, s_8, \cdots s_{2n-1}$.

Now an easy check shows that the $r$-adjacency condition is fulfilled.
Hence, $\chi_r(L(G_n)) = n + 3$, for $r = n + 1 = \Delta$ and $2n \equiv 0 \mod 3$.

**Case 5**: $r = n + 1 = \Delta$ and $2n \equiv 1 \mod 3$

The $r$ dynamic $(n+4)$-coloring is as follows:

- Color the vertex $e_i$ with color $i$, $1 \leq i \leq n$.

It is clear that to color $2n$ remaining vertices: $s_i$ we have to use colors $n, \cdots \chi_r$.
we have to still take care of the $r$-adjacency condition for all vertices.
The $r$-adjacency condition for vertices $s_i, 1 \leq i \leq n$, we must use atleast two new colors to vertex $s_i$.

- Color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-3}$ with color $n+1$.
- Color the vertices $s_2n, s_3, s_6, s_9, s_{12}, \cdots s_{2n-4}$ with new color $n+2$.

Now the vertices $s_2, s_5, s_8, \cdots s_{2n-2}, s_{2n-1}$ are uncolored.
- Color the vertices $s_2, s_5, s_8, \cdots s_{2n-1}$ with the color $n + 3$.

Now $s_{2n-2}$ is uncolored. So we have to assign the new color $n + 4$ to $s_{2n-2}$.

Now an easy check shows that the $r$-adjacency condition is fulfilled.
Hence, $\chi_r(L(G_n)) = n + 4$, for $r = n + 1 = \Delta$ and $2n \equiv 1 \mod 3$.

**Case 6**: $r = n + 1 = \Delta$ and $2n \equiv 2 \mod 3$

The $r$ dynamic $(n+5)$-coloring is as follows:
• color the vertex $e_i$ with color $i$, $1 \leq i \leq n$.

It is clear that to color $2n$ remaining vertices: $s_i$, we have to use colors $n, \cdots, \chi_r$.
we have to still take care of the $r$- adjacency condition for all vertices. The $r$- adjacency condition for vertices $s_i, 1 \leq i \leq n$, we must use atleast two new colors to vertex $s_i$.

• Color the vertices $s_1, s_4, s_7, s_{10}, \cdots s_{2n-4}$ with color $n + 1$.
• color the vertices $s_2, s_3, s_6, s_9, s_{12}, \cdots s_{2n-5}$ with new color $n + 2$.
• color the vertices $s_2, s_5, s_8, \cdots s_{2n-3}$ with the color $n + 3$.

Now $s_{2n-2}, s_{2n-1}$ are uncolored.
So we have to assign the new color $n + 4$ to $s_{2n-2}$ and to assign the new color $n + 5$ to $s_{2n-1}$.
Now an easy check shows that the $r$–adjacency condition is fulfilled. Hence, $\chi_r(L(G_n)) = n + 5$, for $r = n + 1 = \Delta$ and $2n \equiv 2 \mod 3$.

In all cases the order of the assigned colors is important. One can verify that the adjacency and $r$-adjacency conditions are fulfilled.

□

References


