Horadam polynomials and their applications to new family of bi-univalent functions with respect to symmetric conjugate points

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Abstract:

In the current paper, by making use of the Horadam polynomials, we introduce and investigate a new family of holomorphic and biunivalent functions with respect to symmetric conjugate points defined in the open unit disk $D$. We derive upper bounds for the second and third coefficients and solve Fekete-Szegő problem of functions belongs to this family.

Keywords: Bi-univalent function; Horadam polynomials; Upper bounds; Symmetric conjugate; Fekete-Szegő problem; Subordination.


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1. Introduction

Denote by $\mathcal{A}$ the collection of holomorphic functions in the open unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$ that have the form:

\[(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n.\]

Further, let $S$ indicate the sub-collection of $\mathcal{A}$ consisting of functions in $D$ satisfying (1.1) which are univalent in $D$.

Also, let $S_{sc}^*$ be the subclass of $S$ consisting of functions given by (1.1) satisfying

$$\text{Re} \left\{ \frac{zf'(z)}{f(z) - f(-\overline{z})} \right\} > 0, \quad z \in D.$$  

These functions are called starlike with respect to symmetric conjugate points and were introduced by El-Ashwah and Thomas [6]. The class can be extended to other class in $D$, namely convex functions with respect to symmetric conjugate points. Let $C_{sc}$ denote the class of convex functions with respect to symmetric conjugate points and satisfy the conditions

$$\text{Re} \left\{ \frac{(zf'(z))'}{(f(z) - f(-\overline{z}))} \right\} > 0, \quad z \in D.$$  

According to the Koebe One-Quarter Theorem [5] ”every function $f \in S$ has an inverse $f^{-1}$ defined by $f^{-1}(f(z)) = z$, $(z \in D)$ and $f(f^{-1}(w)) = w$, $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})”$, where

$$g(w) = f^{-1}(w) = w-a_2w^2+\left(2a_2^2-a_3\right)w^3-\left(5a_2^3-5a_2a_3+a_4\right)w^4+\cdots.$$  

$$\quad (1.2)$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $D$ if both $f$ and $f^{-1}$ are univalent in $D$. Let $\Sigma$ stands for the class of bi-univalent functions in $D$ given by (1.1). In fact, Srivastava et al. [16] has apparently revived the study of holomorphic and bi-univalent functions in recent years, it was followed by such works as those by Bulut [4], Altinkaya and Yalcın [2, 3], Adegani et al. [1] and others (see, for example [13, 14, 15, 17, 18, 19]).

We notice that the class $\Sigma$ is not empty. For example, the functions $z$, $\frac{z}{1-z}$, $-\log(1-z)$ and $\frac{1+z}{1-z}$ are members of $\Sigma$. However, the Koebe function is not a member of $\Sigma$. Until now, the coefficient estimate problem
for each of the following Taylor-Maclaurin coefficients $|a_n|$, $(n = 3, 4, \cdots)$ for functions $f \in \Sigma$ is still an open problem.

"With a view to recalling the principal of subordination between holomorphic functions, let the functions $f$ and $g$ be holomorphic in $D$. We say that the function $f$ is said to be subordinate to $g$, if there exists a Schwarz function $w$ holomorphic in $D$ with $w(0) = 0$ and $|w(z)| < 1$ ($z \in D$) such that $f(z) = g(w(z))$. This subordination is denoted by $f \prec g$ or $f(z) \prec g(z)$ ($z \in D$). It is well known that (see [12]), if the function $g$ is univalent in $D$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(D) \subset g(D)$".

The Horadam polynomials $h_n(r)$ are defined by the following repetition relation (see [8]):

\begin{equation}
(1.3) \quad h_n(r) = prh_{n-1}(r) +qh_{n-2}(r) \quad (r \in \mathbb{R}, n \in \mathbb{N} = \{1, 2, 3, \cdots \}),
\end{equation}

with $h_1(r) = a$ and $h_2(r) = br$, for some real constant $a, b, p$ and $q$. The characteristic equation of repetition relation (1.3) is $t^2 - prt - q = 0$. This equation has two real roots $x = \frac{pr + \sqrt{p^2r^2 + 4q}}{2}$ and $y = \frac{pr - \sqrt{p^2r^2 + 4q}}{2}$.

**Remark 1.1.** By selecting the particular values of $a, b, p$ and $q$, the Horadam polynomial $h_n(r)$ reduces to several polynomials. Some of them are illustrated below:

1. Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(r)$.

2. Taking $a = 2$ and $b = p = q = 1$, we attain the Lucas polynomials $L_n(r)$.

3. Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(r)$.

4. Taking $a = b = p = 2$ and $q = 1$, we get the Pell-Lucas polynomials $Q_n(r)$.

5. Taking $a = b = 1$, $p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(r)$ of the first kind.

6. Taking $a = 1$, $b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(r)$ of the second kind.
These polynomials, the families of orthogonal polynomials and other special polynomials as well as their generalizations are potentially important in a variety of disciplines in many of sciences, specially in the mathematics, statistics and physics. For more information associated with these polynomials see [7, 8, 10, 11].

The generating function of the Horadam polynomials \( h_n(r) \) (see [9]) is given by

\[
\Pi(r, z) = \sum_{n=0}^{\infty} h_n(r) z^{n-1} = \frac{a + (b - ap)rz}{1 - prz - qz^2}.
\]

2. Main Results

We begin this section by defining the family \( G_\Sigma(\lambda, \eta, r) \) as follows:

**Definition 2.1.** For \( 0 \leq \eta \leq \lambda \leq 1 \) and \( r \in \mathbb{R} \), a function \( f \in \Sigma \) with \( a_n \in \mathbb{R} \) is said to be in the class \( G_\Sigma(\lambda, \eta, r) \) if it fulfills the subordinations:

\[
\frac{2 \left[ \lambda \eta z^3 f'''(z) + (\lambda + \eta(2\lambda - 1)) z^2 f''(z) + zf'(z) \right]}{\lambda \eta z^2 \left( f(z) - \overline{f(-z)} \right)^3 + (\lambda - \eta)z \left( f(z) - \overline{f(-z)} \right) + (1 - \lambda + \eta) \left( f(z) - \overline{f(-z)} \right) - \Pi(r, z) + 1 - a}
\]

and

\[
\frac{2 \left[ \lambda \eta w^3 g'''(w) + (\lambda + \eta(2\lambda - 1)) w^2 g''(w) + wg'(w) \right]}{\lambda \eta w^2 \left( g(w) - \overline{g(-w)} \right)^3 + (\lambda - \eta)w \left( g(w) - \overline{g(-w)} \right) + (1 - \lambda + \eta) \left( g(w) - \overline{g(-w)} \right) - \Pi(r, w) + 1 - a},
\]

where \( a \) is real constant and the function \( g = f^{-1} \) is given by (1.2).

**Theorem 2.1.** For \( 0 \leq \eta \leq \lambda \leq 1 \) and \( r \in \mathbb{R} \), let \( f \in \mathcal{A} \) with \( a_n \in \mathbb{R} \) be in the class \( G_\Sigma(\lambda, \eta, r) \). Then

\[
|a_2| \leq \frac{|br| \sqrt{|br|}}{\sqrt{2 \left[ (6\lambda \eta + 2(\lambda - \eta) + 1) b - 2p (2\lambda \eta + \lambda - \eta + 1)^2 \right] br^2 - 2qa (2\lambda \eta + \lambda - \eta + 1)^2}}
\]

and

\[
|a_3| \leq \frac{|br| \sqrt{b^2 r^2}}{2 (6\lambda \eta + 2(\lambda - \eta) + 1) + \frac{b^2 r^2}{4 (2\lambda \eta + \lambda - \eta + 1)^2}}.
\]
Proof. Let $f \in \mathcal{G}_2(\lambda, \eta, r)$. Then there are two holomorphic functions $u, v : D \rightarrow D$ given by

(2.1) \hspace{1cm} u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \quad (z \in D)

and

(2.2) \hspace{1cm} v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \quad (w \in D),

with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(w)| < 1$, $z, w \in D$ such that

\[
\frac{2 \left[ \lambda \eta z^3 f'''(z) + (\lambda + \eta (2\lambda - 1)) z^2 f''(z) + zf'(z) \right]}{\lambda \eta z^2 \left( f(z) - \overline{f(-z)} \right)'' + (\lambda - \eta) \left( f(z) - \overline{f(-z)} \right)'' + (1 - \lambda + \eta) \left( f(z) - \overline{f(-z)} \right)'} = \Pi(r, u(z)) + 1 - a
\]

and

\[
\frac{2 \left[ \lambda \eta w^3 g'''(w) + (\lambda + \eta (2\lambda - 1)) w^2 g''(w) + wg'(w) \right]}{\lambda \eta w^2 \left( g(w) - \overline{g(-w)} \right)'' + (\lambda - \eta) w \left( g(w) - \overline{g(-w)} \right)'' + (1 - \lambda + \eta) \left( g(w) - \overline{g(-w)} \right)'} = \Pi(r, v(w)) + 1 - a.
\]

Or, equivalently

(2.3) \hspace{1cm} 1 + h_1(r) + h_2(r)u(z) + h_3(r)u^2(z) + \cdots

and

(2.4) \hspace{1cm} 1 + h_1(r) + h_2(r)v(w) + h_3(r)v^2(w) + \cdots.

Combining (2.1), (2.2), (2.3) and (2.4) yields

\[
\frac{2 \left[ \lambda \eta z^3 f'''(z) + (\lambda + \eta (2\lambda - 1)) z^2 f''(z) + zf'(z) \right]}{\lambda \eta z^2 \left( f(z) - \overline{f(-z)} \right)'' + (\lambda - \eta) \left( f(z) - \overline{f(-z)} \right)'' + (1 - \lambda + \eta) \left( f(z) - \overline{f(-z)} \right)'}
\]
\(1 + h_2(r)u_1z + \left[h_2(r)u_2 + h_3(r)u_1^2\right]z^2 + \cdots\)

and

\[
\frac{2 [\lambda w^3 \tilde{g}''(w) + (\lambda + \eta(2\lambda - 1)) w^2 \tilde{g}''(w) + w\tilde{g}'(w)]}{\lambda w^2 \left[g(w) - g(-w)\right]'' + (\lambda - \eta) w \left[g(w) - g(-w)\right]'} + (1 - \lambda + \eta) \left[g(w) - g(-w)\right] = 1 + h_2(r)v_1w + \left[h_2(r)v_2 + h_3(r)v_1^2\right]w^2 + \cdots.
\]

It is quite well-known that if \(|u(z)| < 1\) and \(|v(w)| < 1\), then

\[
|u_i| \leq 1 \quad \text{and} \quad |v_i| \leq 1 \quad \text{for all} \quad i \in \mathbb{N}.
\]

Comparing the corresponding coefficients in (2.5) and (2.6), after simplifying, we have

\[
2(2\lambda \eta + \lambda - \eta + 1) a_2 = h_2(r)u_1,
\]

\[
2(6\lambda \eta + 2(\lambda - \eta) + 1) a_3 = h_2(r)u_2 + h_3(r)u_1^2,
\]

\[
-2(2\lambda \eta + \lambda - \eta + 1) a_2 = h_2(r)v_1
\]

and

\[
2(6\lambda \eta + 2(\lambda - \eta) + 1) \left(2a_2^2 - a_3\right) = h_2(r)v_2 + h_3(r)v_1^2.
\]

In view of (2.8) and (2.10), we conclude that

\[
u_1 = -v_1
\]

and

\[
8(2\lambda \eta + \lambda - \eta + 1)^2 a_2^2 = h_2^2(r)(u_1^2 + v_1^2).
\]

If we add (2.9) to (2.11), we find that

\[
4(6\lambda \eta + 2(\lambda - \eta) + 1) a_2^2 = h_2(r)(u_2 + v_2) + h_3(r)(u_1^2 + v_1^2).
\]

Substituting the value of \(u_1^2 + v_1^2\) from (2.13) into (2.14), it follows that
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\[ a_2^2 = \frac{h_2^3(r)(u_2 + v_2)}{4 \left[ h_2^3(r) (6\lambda\eta + 2(\lambda - \eta) + 1) - 2h_3(r) (2\lambda\eta + \lambda - \eta + 1)^2 \right]} . \]

Further computations using (1.3), (2.7) and (2.15), we deduce that

\[ |a_2| \leq \sqrt{2} \left| \left( 6\lambda\eta + 2(\lambda - \eta) + 1 \right) b - 2p (2\lambda\eta + \lambda - \eta + 1)^2 \right| \frac{br^2 - 2qa (2\lambda\eta + \lambda - \eta + 1)^2}{br^2 - 2qa (2\lambda\eta + \lambda - \eta + 1)^2} . \]

To determinate the bound on \( |a_3| \), by subtracting (2.11) from (2.9), we can easily see that

\[ (6\lambda\eta + 2(\lambda - \eta) + 1) (a_3 - a_2^2) = h_2(r)(u_2 - v_2) + h_3(r)(u_1^2 - v_1^2) . \]

Also, by using (2.12) and (2.13) together with (2.16), we conclude that

\[ a_3 = \frac{h_2(r)(u_2 - v_2)}{4 (6\lambda\eta + 2(\lambda - \eta) + 1)} + \frac{h_3^2(r)(u_1^2 + v_1^2)}{8 (2\lambda\eta + \lambda - \eta + 1)^2} . \]

Thus applying (1.3), we obtain

\[ |a_3| \leq \frac{|br|}{2 (6\lambda\eta + 2(\lambda - \eta) + 1)} + \frac{b^2r^2}{4 (2\lambda\eta + \lambda - \eta + 1)^2} . \]

This completes the proof of Theorem 2.1 \( \square \)

In the next theorem, we discuss the ”Fekete-Szegö problem” for the family \( G_\Sigma(\lambda, \eta, r) \).

**Theorem 2.2.** For \( 0 \leq \eta \leq \lambda \leq 1 \) and \( r, \mu \in \mathbb{R} \), let \( f \in A \) with \( a_n \in \mathbb{R} \) be in the family \( G_\Sigma(\lambda, \eta, r) \). Then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{|br|}{2(6\lambda\eta + 2(\lambda - \eta) + 1)} & \text{for } |\mu - 1| \leq \frac{[6\lambda\eta + 2(\lambda - \eta) + 1)b - 2p(2\lambda\eta + \lambda - \eta + 1)^2]}{br^2 + (6\lambda\eta + 2(\lambda - \eta) + 1)} \, |br|^2 - 2qa(2\lambda\eta + \lambda - \eta + 1)^2, \\
\frac{2[6\lambda\eta + 2(\lambda - \eta) + 1)b - 2p(2\lambda\eta + \lambda - \eta + 1)^2]}{|br|^2 - 2qa(2\lambda\eta + \lambda - \eta + 1)^2} & \text{for } |\mu - 1| \geq \frac{[6\lambda\eta + 2(\lambda - \eta) + 1)b - 2p(2\lambda\eta + \lambda - \eta + 1)^2]}{br^2 + (6\lambda\eta + 2(\lambda - \eta) + 1)} \, |br|^2 - 2qa(2\lambda\eta + \lambda - \eta + 1)^2.
\end{cases}
\]
Proof. In the light of (2.15) and (2.16), we find that
\[
a_3 - \mu a_2^2 = \frac{h_2(r)(u_2 - v_2)}{4(6\lambda\eta + 2(\lambda - \eta) + 1)} + (1 - \mu) a_2^2
\]
\[
= \frac{h_2(r)(u_2 - v_2)}{4(6\lambda\eta + 2(\lambda - \eta) + 1)} + \frac{h_2(r)(u_2 + v_2)(1 - \mu)}{4(h_2(r)(6\lambda\eta + 2(\lambda - \eta) + 1) - 2h_3(r)(2\lambda\eta + \lambda - \eta + 1)^2)}
\]
\[
= \frac{h_2(r)}{4} \left[ \left( \psi(\mu, r) + \frac{1}{6(\lambda\eta + 2(\lambda - \eta) + 1)} u_2 + \left( \psi(\mu, r) - \frac{1}{6(\lambda\eta + 2(\lambda - \eta) + 1)} \right) v_2 \right) \right]
\]
where
\[
\psi(\mu, r) = \frac{h_2(r)(1 - \mu)}{h_2(r)(6\lambda\eta + 2(\lambda - \eta) + 1) - 2h_3(r)(2\lambda\eta + \lambda - \eta + 1)^2}.
\]

According to (1.3), we deduce that
\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|br|}{2(6\lambda\eta + 2(\lambda - \eta) + 1)}, & 0 \leq |\psi(\mu, r)| \leq \frac{1}{6(\lambda\eta + 2(\lambda - \eta) + 1)} \\
\frac{|br|}{2|\psi(\mu, r)|} |\psi(\mu, r)|, & |\psi(\mu, r)| \geq \frac{1}{6(\lambda\eta + 2(\lambda - \eta) + 1)}
\end{array} \right.
\]

After some computations, we obtain
\[
|a_3 - \mu a_2^2| \leq \left\{ \begin{array}{ll}
\frac{|br|}{2(6\lambda\eta + 2(\lambda - \eta) + 1)} & \text{for } |\mu - 1| \leq \frac{|(6\lambda\eta + 2(\lambda - \eta) + 1)b - 2\mu(2\lambda\eta + \lambda - \eta + 1)^2|}{b^2r^2(6\lambda\eta + 2(\lambda - \eta) + 1)} \\
\frac{|br|^3}{|\mu - 1|} & \text{for } |\mu - 1| \geq \frac{|(6\lambda\eta + 2(\lambda - \eta) + 1)b - 2\mu(2\lambda\eta + \lambda - \eta + 1)^2|}{b^2r^2(6\lambda\eta + 2(\lambda - \eta) + 1)}
\end{array} \right.
\]

Putting $\mu = 1$ in Theorem 2.2, we obtain the following result:

**Corollary 2.1.** For $0 \leq \eta \leq \lambda \leq 1$ and $r \in \mathbb{R}$, let $f \in A$ be in the family $G_\Sigma(\lambda, \eta, r)$. Then
\[
|a_3 - a_2^2| \leq \frac{|br|}{2(6\lambda\eta + 2(\lambda - \eta) + 1)}.
\]
References


