H-supplemented modules with respect to images of fully invariant submodules

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Abstract:
Lifting modules play important roles in module theory. \(H\)-supplemented modules are a nice generalization of lifting modules which have been studied extensively recently. In this article, we introduce a proper generalization of \(H\)-supplemented modules via images of fully invariant submodules. Let \(F\) be a fully invariant submodule of a right \(R\) module \(M\). We say that \(M\) is \(IF - H\)-supplemented in case for every endomorphism \(\varphi\) of \(M\), there is a direct summand \(D\) of \(M\) such that \(\varphi(\langle F \rangle) + X = M\) if and only if \(D + X = M\), for every submodule \(X\) of \(M\). It is proved that \(M\) is \(I_{r}\) -\(H\)-supplemented if and only if \(F\) is a dual Rickart direct summand of \(M\) for a fully invariant noncosingular submodule \(F\) of \(M\). It is shown that the direct sum of \(I_{r}\) -\(H\)-supplemented modules is not in general \(I_{r}\) -\(H\)-supplemented. Some sufficient conditions such that the direct sum of \(I_{r}\) -\(H\)-supplemented modules is \(I_{r}\) -\(H\)-supplemented are given.

Keywords: \(H\)-supplemented module; \(I_{r}\) -lifting module; \(I_{r}\) -\(H\)-supplemented module; Dual Rickart module; Endomorphisms ring.


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1. Introduction

All rings considered in this article are associative with an identity element and all modules are unitary right modules unless otherwise stated. Let $R$ be a ring and $M$ a right $R$-module. The ring of all $R$-endomorphisms of $M$ is denoted by $S = \text{End}_R(M)$. We use the notation $N \ll M$ to indicate that $N$ is small in $M$ (i.e., for all $L \subsetneq M$, $L + N \neq M$). A module $M$ is called hollow if every proper submodule of $M$ is small in $M$. $\text{Rad}(M)$ and $\text{Soc}(M)$ denote the radical and the socle of a module $M$, respectively. A submodule $N$ of $M$ is called a fully invariant submodule of $M$ if for all $\phi \in \text{End}_R(M)$, $\phi(N) \subseteq N$.

Let $L \subseteq K \leq M$. We recall that $K$ lies above $L$ in $M$, if $K/L \ll M/L$. A module $M$ is called lifting if every submodule $A$ of $M$ lies above a direct summand $D$ of $M$. A submodule $N$ of $M$ is called supplement in $M$ if there exists a submodule $K$ of $M$ such that $M = N + K$ and $N \cap K \ll N$. A module $M$ is said to be supplemented if every submodule of $M$ has a supplement [4].

Recall that a module $M$ is called $H$-supplemented in case for every submodule $N$ of $M$, there exists a direct summand $D$ of $M$ such that $M = N + X$ if and only if $M = D + X$ for every submodule $X$ of $M$ [9]. In [7], the authors presented some equivalent conditions for a module to be $H$-supplemented that shows that this class of modules is closely related to the concept of small submodules. In [3], the authors introduced a new generalization of $H$-supplemented modules that is Goldie*-supplemented modules via an equivalence relation namely $\beta^*$. Let $X$ and $Y$ be submodules of $M$. Then $X\beta^*Y$ in $M$ provided $(X + Y)/X \ll M/X$ and $(X + Y)/Y \ll M/Y$. Here it is convenient to state that $M$ is $H$-supplemented if and only if for each submodule $X$ of $M$ there exists a direct summand $D$ of $M$ such that $X\beta^*D$ in $M$.

Recall from [8], that a module $M$ is dual Rickart in case, for every endomorphism $\varphi$ of $M$, the image of $\varphi$ is a direct summand of $M$. The author in [1] introduced a generalization of both lifting modules and dual Rickart modules as $I$-lifting modules. The author showed that a projective $I$-lifting module is a direct sum of cyclic modules. In [10], the authors studied $H$-supplemented modules via homomorphisms, which generalizes both $H$-supplemented modules and $I$-lifting modules. They called a module $M$, endomorphism $H$-supplemented ($E$-$H$-supplemented, for short) provided for every $\varphi$ in $\text{End}_R(M)$, there is a direct summand $D$ of $M$ such that $\text{Im}\varphi \beta^*D$. 
The present authors, in [2] introduced a new generalization of $\mathcal{I}$-lifting modules via image of fully invariant submodules. Let $M$ be a module and let $F$ be a fully invariant submodule of $M$. Then $M$ is called $\mathcal{I}_F$-lifting in case for every endomorphism $\varphi$ of $M$, there is a direct summand $D$ of $M$ contained in $\varphi(F)$ such that $\varphi(F)/D \ll M/D$. Some properties of $\mathcal{I}_F$-lifting modules were investigated in [2].

Inspired by mentioned works on lifting modules and $H$-supplemented modules via a homological approach, we are interested to study on $H$-supplemented modules via image of fully invariant submodules. In fact, in the definition of an $E$-$H$-supplemented module, one can replaced $M$ by a fully invariant submodule of $M$. Let $M$ be a module and let $F$ be a fully invariant submodule of $M$. We say $M$ is $\mathcal{I}_F$-$H$-supplemented provided for every endomorphism $\varphi$ of $M$ there is a direct summand $D$ of $M$ such that $\varphi(F) + X = M$ if and only if $D + X = M$, equivalently $\varphi(F)\beta^*D$. In what follows by $F$ we mean a fully invariant submodule of $M$.

One preference of this generalization of the $H$-supplemented modules over other generalizations is that the fully invariant submodules form a complete modular sublattice of the lattice of submodules and they are well mannered with respect to endomorphisms. Many of the important submodules of a module are fully invariant submodules such as the Jacobson radical of a module, the socle of a module, the singular submodule, the cosingular submodule, etc.

In Section 2, we show that $\mathcal{I}_F$-$H$-supplemented modules are proper generalization of both $\mathcal{I}_F$-lifting modules and $H$-supplemented modules. Examples are provided to show that the concept of an $\mathcal{I}_F$-$H$-supplemented module is distinct from both an $\mathcal{I}_F$-lifting module and an $H$-supplemented module. As we state in the introduction, we provide conditions under which

2. $\mathcal{I}_F$-$H$-supplemented modules

In this section we introduce $\mathcal{I}_F$-$H$-supplemented modules as a proper generalization of both $\mathcal{I}_F$-lifting modules and $H$-supplemented modules. Examples are provided to show that the concept of an $\mathcal{I}_F$-$H$-supplemented module is distinct from both an $\mathcal{I}_F$-lifting module and an $H$-supplemented module. As we state in the introduction, we provide conditions under which
the two concepts of \( I_F \)-lifting and \( I_F \)-supplemented coincide.

**Definition 2.1.** Let \( M \) be a module and let \( F \) be a fully invariant submodule of \( M \). We say \( M \) is \( I_F \)-H-supplemented if for every \( \varphi \in \text{End}_R(M) \), there exists a direct summand \( D \) of \( M \) such that \( \varphi(F) + X = M \) if and only if \( D + X = M \) for every submodule \( X \) of \( M \).

It is clear that every \( H \)-supplemented module is \( I_F \)-H-supplemented but the converse is not true (see Example 2.12). Obviously, the sentences “\( M \) is \( E \)-H-supplemented” and “\( M \) is \( I_M \)-H-supplemented” are the same.

Note that every module \( M \) is clearly \( I_0 \)-H-supplemented.

It is proved in [7] that a module \( M \) is \( H \)-supplemented if and only if for every submodule \( N \) of \( M \) there is a direct summand \( D \) of \( M \) such that \((N + D)/D \ll M/D\) and \((N + D)/N \ll M/N\), i.e., \( N \beta^* D \). The same is true for \( I_F \)-H-supplemented modules. One can easily check the following:

**Proposition 2.2.** The following sentences are equivalent for a module \( M \):
1. \( M \) is \( I_F \)-H-supplemented;
2. For every \( \varphi \in S \), there exists a direct summand \( D \) of \( M \) such that \( \varphi(F) \beta* D \);
3. For every \( \varphi \in S \), there exist a direct summand \( D \) and a submodule \( N \) of \( M \) with \( \varphi(F) \subseteq N \) and \( D \subseteq N \) such that \( N \beta D \ll M \beta \) and \( N \varphi(F) \ll M \varphi(F) \).

Below, we shall provide some examples of \( I_F \)-H-supplemented modules.

**Examples 2.3.**
1. Every \( I_F \)-lifting module is \( I_F \)-H-supplemented. In particular every lifting module \( M \) is \( I_F \)-H-supplemented for every fully invariant submodule \( F \) of \( M \).
2. Let \( p \) be a prime number. Then the \( \mathbb{Z} \)-module \( M = \mathbb{Z}_{p^2} \) is not a dual Rickart module. Now, \( \text{Rad}(M) = (p) \neq 0 \). Since \( M \) is a hollow module, \( M \) is \( I_{\text{Rad}(M)} \)-lifting and hence \( I_{\text{Rad}(M)} \)-H-supplemented.

A module \( M \) is called **epi-retractable** provided every submodule of \( M \) is a homomorphic image of \( M \) [5]. By [5, Example 2.4], every finitely generated module over an \( \text{PID} \) is epi-retractable. Note that for an epi-retractable module the two concepts \( H \)-supplemented and \( E \)-H-supplemented coincide. Now it is easy to verify the following proposition:

**Proposition 2.4.** Let \( F \) be a fully invariant submodule of an epi-retractable module \( M \). If \( M \) is \( E \)-H-supplemented, then \( M \) is \( I_F \)-H-supplemented.
We show that the class of $IF$-$H$-supplemented modules contains properly the class of $IF$-lifting modules.

**Examples 2.5.** (1) Let $p$ be a prime number. Consider the $\mathbb{Z}$-module $M_1 = \mathbb{Z}_{p^3}$. Then by [7, Example 4.6], the $\mathbb{Z}$-module $M = M_1 \oplus \frac{M_1}{(p)} \oplus \frac{(p^2)}{(p^2)}$ is $H$-supplemented. Since $\mathbb{Z}_p \oplus \mathbb{Z}_{p^3}$ is isomorphic to a direct summand of $M$, $M$ is not lifting from [6, Corollary 2]. Being $M$ a finitely generated $\mathbb{Z}$-module implies that $M$  is epi-retractable by [5, Example 2.4]. Hence $M$ is not $IF$-lifting which means that $M$ is not $IM$-lifting. In other words, $M$ is $IM$-$H$-supplemented as well as $H$-supplemented.

(2) (see [13, Example 2.3]) Let $I$ and $J$ be two ideals of a commutative local ring $R$ with maximal ideal $m$ such that $I \subset J \subseteq m$ and $mJI$ (e.g., $R$ is an DVR with maximal ideal $m$, $I = m^4$ and $J = m^2$). We consider the module $M = R/I \times R/J$. From [13, Proposition 2.1] it follows that $M$ is $H$-supplemented and so $M$ is $IM$-$H$-supplemented. In other words, from [13, Example 2.3], $M$ is not lifting. Being $M$ an epi-retractable module implies $M$ is not $I$-lifting ($IM$-lifting).

Recall from [12] that an $R$-module $M$ is *noncosingular* (*cosingular*) provided $\overline{Z}(M) = M$ ($\overline{Z}(M) = 0$) where $\overline{Z}(M) = \cap\{\text{Ker} f \mid f: M \rightarrow U\}$ for all small $R$-modules $U$.

We present some conditions under which, the two concepts $IF$-lifting and $IF$-$H$-supplemented coincide.

**Theorem 2.6.** Let $F$ be a fully invariant submodule of a module $M$. If either $F$ is noncosingular or $\text{Rad}(M) = 0$, then the following statements are equivalent:

1. For every $\varphi \in \text{End}_R(M)$, the submodule $\varphi(F)$ is a direct summand of $M$;
2. $M$ is $IF$-lifting;
3. $M$ is $IF$-$H$-supplemented;
4. $F$ is a dual Rickart direct summand of $M$.

**Proof.** We prove when $F$ is noncosingular, the case $\text{Rad}(M) = 0$ is the same.

1) $\Rightarrow$ 2) It is obvious.

2) $\Rightarrow$ 3) It can be easily verified.

3) $\Rightarrow$ 4) Let $M$ be $IF$-$H$-supplemented and let $\varphi$ be an endomorphism of $M$. Then there is a direct summand $D$ of $M$ such that $\varphi(F)\beta^*D$ which means that $(\varphi(F) + D)/D \ll M/D$ and $(\varphi(F) + D)/\varphi(F) \ll M/\varphi(F)$. 
Note that $\varphi(F)$ is noncosingular as well as $F$. It follows that $(\varphi(F) + D)/D \cong \varphi(F)/(\varphi(F) \cap D)$ is a noncosingular submodule of $M/D$. Hence, $\varphi(F) + D = D$ implies that $\varphi(F) \subseteq D$. Now, $D/\varphi(F) \ll M/\varphi(F)$. Set $D \oplus D' = M$. Then $D/\varphi(F) + (D' + \varphi(F))/\varphi(F) = M/\varphi(F)$. Therefore, $D' + \varphi(F) = M$. Being $\varphi(F)$ a submodule of $D$ combining with modularity implies $\varphi(F) = D$. Now, let $\psi$ be an endomorphism of $F$. Then $h = j_0 \psi_0 \pi_F$ is an endomorphism of $M$ where $j$ is the inclusion map and $\pi_F$ is the canonical projection. Then $\psi(F) = h(F)$ is a direct summand of $M$. Hence $\psi(F)$ is a direct summand of $F$ showing that $F$ is dual Rickart.

(4) $\Rightarrow$ (1) Let $\varphi$ be an arbitrary endomorphism of $M$. Then $g = \pi_F \circ \varphi \circ j$ is an endomorphism of $F$. As $F$ is a dual Rickart module, $g(F)$ is a direct summand of $F$ and also a direct summand of $M$. It follows that $\varphi(F) = g(F)$ is a direct summand of $M$. $\square$

**Example 2.7.** [2, Example 2.8] (1) Let $K$ be a field and $R = \prod_{i=1}^{\infty} K_i$ where $K_i = K$ for each $i \in \mathbb{N}$.

Let $L$ be an $V$-ring and let $K$ be a field. Then $S = K \times L$ is an $V$-ring as well. Consider the central idempotent $e = (1,0)$ of $S$. Then $Se = eS \cong K$ as both left and right $S$-module. Let $R$ be the ring $M_n(S)$ (the ring of all $n \times n$ matrices with entries from $S$). As $R$ is Morita-equivalent to $S$, it should be also an $V$-ring. Now, $R$ has a central idempotent, $f = eI$ where $I$ is the identity matrix of $R$. Then $fR = Rf$ is isomorphic to $M_n(Se)$ so that $fR = Rf \cong M_n(K)$. Note that $F = Rf$ is a two-sided ideal of $R$ and also is a direct summand of $R$. Being $K$ a field implies that $M_n(K)$ and hence $F$ is semisimple (dual Rickart). It follows from Theorem 2.6 that $R$ is an $R-F-H$-supplemented module.

Recall from [11] that a module $M$ is *weak duo* in case every direct summand of $M$ is a fully invariant submodule of $M$. We recall that $L$ is a *cosmall submodule of $K$ in $M$* (denoted by $L \triangleleft_{cs} K$ in $M$) if $K$ lies above $L$ in $M$. Recall that a submodule $L$ of $M$ is called *coclosed* in $M$, if $L$ has no proper cosmall submodule. It is clear that every direct summand of $M$ is a coclosed submodule of $M$. A module $M$ is said to have *cosmall intersection property or CSIP* if for any $A, B, C, D \subseteq M$, $A \triangleleft_{cs} B$ in $M$ and $C \triangleleft_{cs} D$ in $M$ imply that $A \cap C \triangleleft_{cs} B \cap D$ in $M$.

**Proposition 2.8.** Let $F$ be a fully invariant submodule of a module $M$. If either

(1) $M$ is noncosingular or
(2) $M$ is a weak duo module or
(3) $M$ is projective or
(4) $M$ has CSIP,
then the two concepts $I_F$-H-supplemented and $I_F$-lifting coincide.

**Proof.** (1) Let $M$ be $I_F$-H-supplemented and let $\varphi \in \text{End}_R(M)$ be arbitrary. Then by assumption, there is a direct summand $D$ of $M$ such that $(\varphi(F) + D)/D \leq M/D$ and $(\varphi(F) + D)/\varphi(F) \leq M/\varphi(F)$. Since $M$ is noncosingular, $D$ is noncosingular. It follows that $(\varphi(F) + D)/\varphi(F) = 0$ which implies that $\varphi(F) + D = \varphi(F)$. So, $D$ is contained in $\varphi(F)$. Therefore, $\varphi(F)/D \leq M/D$. The converse is straightforward.

(2) Let $M$ be an $I_F$-H-supplemented weak duo module and $\varphi \in \text{End}_R(M)$. Then there exists a direct summand $D$ of $M$ such that $\varphi(F) + X = M$ if and only if $D + X = M$ for every $X \leq M$. Set $D \oplus D' = M$. Then $\varphi(F) + D' = M$. As $F$ is fully invariant we have $F = (F \cap D) \oplus (F \cap D')$. It follows that $M = \varphi((F \cap D) + (F \cap D')) + D' = \varphi(F \cap D) \oplus D'$ (note that $D'$ is also fully invariant so that $\varphi(F \cap D') \subseteq D'$). Hence $\varphi(F \cap D) = D$ implies that $D$ is contained in $\varphi(F)$. Suppose that $\varphi(F)/D + L/D = M/D$ for a submodule $L$ of $M$ containing $D$. Then $\varphi(F) + L = M$. Hence $D + L = M$ which implies $L = M$ as required. Therefore, $\varphi(F)/D \leq M/D$.

(3) Similar to the proof of [10, Theorem 2.16].

(4) Let $\phi \in \text{End}_R(M)$. Then there exists a direct summand $D$ of $M$ such that $\phi(F) \cong \phi(F) + D$ in $M$ and $D \cong \phi(F) + D$ in $M$. By CSIP, $\phi(F) \cap D \cong \phi(F) + D$ in $M$. But $\phi(F) \cap D \leq D \leq \phi(F) + D$, so $\phi(F) \cap D \cong \phi(F) + D$ in $M$. As $D$ is coclosed in $M$, $\phi(F) \cap D = D$, hence $D \leq \phi(F)$ and $D \cong \phi(F)$ in $M$. Therefore $M$ is $I_F$-lifting. □

A characterization of indecomposable $I_F$-H-supplemented modules is presented in the following.

**Proposition 2.9.** Let $M \neq 0$ be an indecomposable module and let $F < M$ be fully invariant. Then $M$ is $I_F$-H-supplemented if and only if $F \ll M$.

In case $F = M$, then $M$ is $I_M$-H-supplemented if and only if every nonzero endomorphism $\varphi$ of $M$ is epimorphism or $\text{Im}\varphi \ll M$.

**Proof.** Let $M$ be $I_F$-H-supplemented and let $i \in \text{End}_R(M)$ be the identity endomorphism. Then there exists a direct summand $D$ of $M$ such that $F + X = i(F) + X = M$ if and only if $D + X = M$ for every submodule $X$ of $M$. By assumption, either $D = 0$ or $D = M$. Second case will not happen as $F < M$. On the other hand, $D = 0$ implies that $F \ll M$. 

...
The converse is obvious as for every \( \varphi \in \text{End}_R(M) \), the condition \( F \ll M \) implies \( \varphi(F) + X = M \) if and only if \( 0 + X = M \), for every \( X \leq M \). The latter follows from [10, Proposition 2.3]. \( \square \)

Following presents a characterization of an \( \mathcal{I}_F\)-\( H \)-supplemented module \( M \) when \( F \) is a direct summand of \( M \).

**Theorem 2.10.** Let \( F \) be a fully invariant direct summand of a module \( M \). If \( F \) is \( E-H \)-supplemented, then \( M \) is \( \mathcal{I}_F\)-\( H \)-supplemented. The converse holds, in case \( M \) is a weak duo module.

**Proof.** (\( \Rightarrow \)) Let \( F \) be \( E-H \)-supplemented and let \( \varphi \) be an endomorphism of \( M \). Consider \( q = \pi_F \circ \varphi \circ \pi_F : F \to F \), which is an endomorphism of \( F \), where \( \pi_F : F \to M \) is the inclusion and \( \pi_F : M \to F \) is the projection map on \( F \). Being \( F \) a fully invariant submodule of \( M \) implies that \( q(F) = \varphi(F) \).

As \( F \) is \( E-H \)-supplemented, there is a direct summand \( D \) of \( F \) (so that of \( M \)) such that \( q(F) + Y = F \) if and only if \( D + Y = F \) for every submodule \( Y \) of \( F \). Now, suppose that \( \varphi(F) + X = M \) for a submodule \( X \) of \( M \). Then \( \varphi(F) + X \cap F = F \). Then, \( D + (X \cap F) = F \). It follows that \( D + X = F + X = M \). For the converse, let \( D + X = M \) where \( X \leq M \). Then modularity implies \( D + (X \cap F) = F \). Since \( F \) is \( E-H \)-supplemented, \( \varphi(F) + (X \cap F) = F \). Hence \( \varphi(F) + X = F + X = M \) as required. Therefore, \( M \) is \( \mathcal{I}_F\)-\( H \)-supplemented.

(\( \Leftarrow \)) Let \( g : F \to F \) be an endomorphism of \( F \) and \( F \oplus F' = M \) for a submodule \( F' \) of \( M \). Then \( h = \varphi \circ \pi_F : M \to M \) is an endomorphism of \( M \) where \( \pi_F : F \to M \) is the inclusion and \( \pi_F : M \to F \) is the projection on \( F \). It is straightforward to check \( h(F) = g(F) \). As \( M \) is \( \mathcal{I}_F\)-\( H \)-supplemented, there exists a direct summand \( D \) of \( M \) such that \( g(F) + X = M \) if and only if \( D + X = M \) for every submodule \( X \) of \( M \). We shall verify \( g(F) + Y = F \) if and only if \( (F \cap D) + Y = F \). Now, let \( g(F) + Y = F \) for \( Y \leq F \). Then \( g(F) + Y + F' = M \). By assumption we have \( D + Y + F' = M \). By modular law, we conclude that \( Y + (D + F') \cap F = F \). Since \( M \) is weak duo, \( D \) is a fully invariant submodule of \( M \) so that \( D = (D \cap F) \oplus (D \cap F') \). Therefore, \( Y + [(D \cap F) + F'] \cap F = F \). Hence \( Y + (D \cap F) = F \). The other implication can be verified similarly. \( \square \)

**Corollary 2.11.** (1) Let \( M \) be a module such that \( \mathcal{Z}(M) \) is a direct summand of \( M \). If \( \mathcal{Z}(M) \) is \( E-H \)-supplemented, then \( \varphi(\mathcal{Z}(M)) \) is a direct summand of \( M \) for every \( \varphi \in \text{End}_R(M) \).
(2) Let $M$ be a module such that $\text{Soc}(M)$ is a direct summand of $M$. Then $M$ is $\mathcal{I}_{\text{Soc}(M)}$-$H$-supplemented.

**Proof.** (1) Let $\mathcal{Z}(M)$ be an $E$-$H$-supplemented direct summand of $M$. Then by Theorem 2.10, $M$ is $\mathcal{I}_{\mathcal{Z}(M)}$-$H$-supplemented. Note that since $\mathcal{Z}(M)$ is a direct summand of $M$, it is noncosingular. Therefore, $\varphi(\mathcal{Z}(M))$ is a direct summand of $M$ for every $\varphi \in \text{End}_{R}(M)$ by Theorem 2.6. (2) It is clear as $\text{Soc}(M)$ is semisimple. □

The following example introduces an $\mathcal{I}_{F}$-$H$-supplemented module which is not $H$-supplemented.

**Example 2.12.** Let $K$ be a field and $R = K \times K[[x]]$. Then $J(R) = 0 \times (x)$. It follows that $R/J(R) \cong K \times (K[[x]])/(x)$ is semisimple. Hence $R$ is a commutative semilocal ring with $\text{Soc}(R) = K \times 0$. Let $M = R^{(N)}_{\mathcal{R}}$. Then $\text{Rad}(R^{(N)}_{\mathcal{R}})$ is not small in $R^{(N)}_{\mathcal{R}}$ by [14, 43.9]. Hence, by [14, 42.5], $M = R^{(N)}_{\mathcal{R}}$ is not supplemented. So $M$ is not $H$-supplemented. By [2, Proposition 2.13], $M$ is not $\mathcal{I}_{\text{Rad}(M)}$-lifting and it is not $\mathcal{I}_{\text{Rad}(M)}$-$H$-supplemented, by Proposition 2.8(3).

Since $\text{Soc}(M) = K^{(N)}_{\mathcal{R}}$ is a direct summand of $M$, by Corollary 2.11, $M$ is $\mathcal{I}_{K^{(N)}_{\mathcal{R}}}$-$H$-supplemented. Therefore, by Proposition 2.8, $M$ is also $\mathcal{I}_{K^{(N)}_{\mathcal{R}}}$-lifting.

**Proposition 2.13.** Let $F$ be a fully invariant submodule of a module $M$ and let $K$ be a fully invariant direct summand of $M$ contained in $F$. If $M$ is $\mathcal{I}_{F}$-$H$-supplemented, then $M/K$ is $\mathcal{I}_{F/K}$-$H$-supplemented.

**Proof.** Let $g : M/K \to M/K$ be an endomorphism of $M/K$ and $M = K \oplus K'$. Then $f = \text{johogo}\pi : M \to M$ is an endomorphism of $M$. Note that $\pi : M \to M/K$ is the canonical projection, $h : M/K \to K'$ is the isomorphism induced by the decomposition $M = K \oplus K'$ and $j : K' \to M$ is the inclusion. By assuming $g(F/K) = T/K$, one can $f(F) = T \cap K'$. Since $M$ is $\mathcal{I}_{F}$-$H$-supplemented, there is a direct summand $D$ of $M$ such that $(T \cap K') + X = M$ if and only if $D + X = M$ for every $X \subseteq M$. Set $M = D \oplus D'$. Then $M/K = (D + K)/K + (D' + K)/K$. Since $K$ is a fully invariant submodule of $M$, we have $K = (K \cap D) \oplus (K \cap D')$. Hence $(D + K) \cap (D' + K) = K$ which implies that $(D + K)/K$ is a direct summand of $M/K$. We shall show that $T/K + Y/K = M/K$ if and only if $(D + K)/K + Y/K = M/K$ for every submodule $Y$ of $M$ containing $K$. In
first step, let $T/K + Y/K = M/K$. Then $T + Y = M$. As $T$ contains $K$, we have \([K + (T \cap K')] + Y = M = (T \cap K') + Y = M\). Then by assumption, $D + Y = M$. It follows that $(D + K)/K + Y/K = M/K$. In other words, suppose that $(D + K)/K + Y/K = M/K$. Then $D + Y = M$ which implies that $T + Y = M$. Therefore, $T/K + Y/K = M/K$ as required. \(\square\)

2. Direct sums of $\mathcal{I}_F$-H-supplemented modules

Let $F = \bigoplus_{i \in I} F_i$ where $F_i \ (i \in I)$ is a fully invariant submodule of $M$. The following example shows that a finite direct sum of $\mathcal{I}_F$-H-supplemented modules need not be $\mathcal{I}_F$-H-supplemented, in general.

Example 3.1. Let $R$ be a discrete valuation ring and let $I_1, \ldots, I_n$ be some ideals of $R$. Consider the $R$-module $M \cong R/I_1 \times \cdots \times R/I_n$. Since $R$ is commutative, each $R/I_i$ is $\mathcal{H}$-supplemented and so each $R/I_i$ is $\mathcal{I}_R/I_i$-H-supplemented. If $I_1 \subseteq \cdots \subseteq I_n \subset R$, then $M$ is $\mathcal{H}$-supplemented by [13, Proposition 2.1]. Therefore, $M$ is $\mathcal{I}_M$-H-supplemented. Otherwise, i.e., the condition $I_1 \subseteq \cdots \subseteq I_n \subset R$ does not hold, $M$ is not $\mathcal{H}$-supplemented. Note also that $M$ is an epi-retractable $R$-module by [5, Example 2.4(3)]. It means that in this case $M$ is not $\mathcal{I}_M$-H-supplemented.

Now we define relative $\mathcal{I}_F$-H-supplemented modules and we apply this concept to study finite direct sums of $\mathcal{I}_F$-H-supplemented modules.

Definition 3.2. Let $M$ and $N$ be $R$-modules and let $F$ be a fully invariant submodule of $M$. We say $M$ is $N$-$\mathcal{I}_F$-H-supplemented if for every homomorphism $\phi: M \to N$, there exists a direct summand $D$ of $N$ such that $\phi(F) + X = N$ if and only if $D + X = N$ for every submodule $X$ of $N$.

It is clear that a module $M$ is $\mathcal{I}_F$-H-supplemented if and only if $M$ is $M$-$\mathcal{I}_F$-H-supplemented.

Theorem 3.3. Let $M$ and $N$ be right $R$-modules and let $F$ be a fully invariant submodule of $M$. Then $M$ is $N$-$\mathcal{I}_F$-H-supplemented if and only if for every direct summand $M'$ of $M$ and every fully invariant direct summand $N'$ of $N$, $M'$ is $N'$-$\mathcal{I}_{F \cap M'}/M'$-H-supplemented.

Proof. Let $M' = eM$ for some $e^2 = e \in \text{End}_R(M)$, and let $N'$ be a fully invariant direct summand of $N$. Then $N = N' \oplus T$ for some $T \leq N$. Suppose that $\psi \in \text{Hom}(M', N')$. We want to show that for any submodule
X of N', there exists a direct summand D of N' such that \( \psi(F \cap M') + X = N' \) if and only if \( D + X = N' \).

First note that \( e(F \cap M') = eF \) and \( \psi e(F) = \psi(F \cap M') \). Let \( \psi(F \cap M') + X = N' \) for a submodule X of N'. Then \( \psi e(F) + X + T = N \). Since \( \psi e M \subseteq N' \subseteq N \) and M is N-I-F-H-supplemented, we conclude that \( A + X + T = N \) for some direct summands A of N. So \( X + (A \cap N') = N' \).

Note that \( A \cap N' \) is a direct summand of \( N' \) as \( N' \) is fully invariant.

Conversely, assume that \( N' = A \cap N' + X \) where \( A \cap N' \) is a direct summand of \( N' \) and \( X \leq N' \). Then \( N = N' + T = A \cap N' + X + T \). Since \( A \cap N' \) is a direct summand of N and M is N-I-F-H-supplemented, \( N = \psi e(F) + X + T \). Hence \( N' = \psi e(F) + X \), and so \( N' = \psi(F \cap M') + X \). Therefore \( M' \) is N-I-F\( \cap M' \)-H-supplemented. The other side of this theorem is clear. \( \Box \)

**Corollary 3.4.** Let \( M \) be R-module and let \( F \) be a fully invariant submodule of \( M \). Then the following condition are equivalent: (1) \( M \) is I-F-H-supplemented; (2) For any fully invariant direct summand \( N \) of \( M \), every direct summand \( L \) of \( M \) is N-I-F\( \cap L \)-H-supplemented.

**Corollary 3.5.** Let \( F \) be a fully invariant submodule of a module \( M \) and let \( L \) be a direct summand of \( M \). If \( M \) is I-F-H-supplemented, then \( L \) is I-F\( \cap L \)-H-supplemented.

Recall that a module \( M \) is said to have the summand sum property (SSP) if the sum of any two direct summands is a direct summand of \( M \).

**Theorem 3.6.** Let \( M = \bigoplus_{i=1}^{n} M_i \) and \( N \) be right R-modules and let \( F \) be a fully invariant submodule of \( M \). If \( N \) has the SSP, then \( M = \bigoplus_{i=1}^{n} M_i \) is N-I-F-H-supplemented if and only if \( M_i \) is N-I-F\( \cap M_i \)-H-supplemented for all \( i \in \{1, 2, \ldots, n\} \).

**Proof.** Assume that \( M \) is N-I-F-H-supplemented. By Theorem 3.3, \( M_i \) is N-I-F\( \cap M_i \)-H-supplemented for all \( i \in \{1, 2, \ldots, n\} \). Conversely, let \( \phi \) be a homomorphism from \( M \) to \( N \). Consider \( \phi = (\phi_i)_{i=1}^{n} \) where \( \phi_i \in \text{Hom}_R(M_i, N) \) and \( i \in \{1, 2, \ldots, n\} \). Since \( M_i \) is N-I-F\( \cap M_i \)-H-supplemented, there exists a direct summand \( D_i \) of \( N \) such that \( \phi_i(F \cap M_i) \beta^* D_i \) for all \( i \in \{1, 2, \ldots, n\} \). Using [3, Proposition 2.11] and this fact that \( \phi(F) = \sum_{i=1}^{n} \phi_i(F \cap M_i) \), we conclude that \( \phi(F) \beta^* \sum_{i=1}^{n} D_i \). As \( N \) has the SSP, \( \sum_{i=1}^{n} D_i \) is a direct summand of \( N \). Hence \( M \) is N-I-F-H-supplemented. \( \Box \)
Proposition 3.7. Let $M = M_1 \oplus M_2$ be a module, where $M_1$ and $M_2$ are fully invariant submodules of $M$. If $M$ is $\mathcal{I}_{M_i}$-supplemented for $i = 1, 2$, and $M$ has SSP, then $M$ is $\mathcal{I}_{M}$-supplemented.

Proof. Let $\phi \in \text{End}_R(M)$. Then, by assumption, there are direct summands $D_1$ and $D_2$ of $M$ such that $\phi(M_1)\beta^*D_1$ and $\phi(M_2)\beta^*D_2$. Note that, since $M$ has SSP, then $D_1 + D_2$ is a direct summand of $M$. By [3, Proposition 2.11], $\phi(M)\beta^*D_1 + D_2$. Therefore $M$ is $\mathcal{I}_{M}$-supplemented.

$\square$

Theorem 3.8. Let $M = M_1 \oplus M_2$ be a duo module and let $F$ be a submodule of $M$. Then $M$ is $\mathcal{I}_{F}$-supplemented if and only if $M_i$ is $\mathcal{I}_{F \cap M_i}$-supplemented for $i = 1, 2$.

Proof. $(\Rightarrow)$ By Corollary 3.5. $(\Leftarrow)$ Assume $M_1$ is $\mathcal{I}_{F \cap M_1}$-supplemented and $M_2$ is $\mathcal{I}_{F \cap M_2}$-supplemented. Let $\pi_i$ be the projection of $M$ on $M_i$ and let $j_i$ be the inclusion map from $M_i$ to $M$ for $i = 1, 2$. Assume that $f$ is an endomorphism of $M$. Since $M_1$ is $\mathcal{I}_{F \cap M_1}$-supplemented, there exists a direct summand $N_1$ of $M_1$ such that $M_1 = N_1 + X$ if and only if $M_1 = \pi_1 f j_1 (F \cap M_1) + X$ for any submodule $X$ of $M_1$ and since $M_2$ is $\mathcal{I}_{F \cap M_2}$-supplemented, there exists a direct summand $N_2$ of $M_2$ such that $M_2 = N_2 + Y$ if and only if $M_2 = \pi_2 f j_2 (F \cap M_2) + Y$ for any submodule $Y$ of $M_2$. We claim that $M = N_1 \oplus N_2 + Z$ if and only if $M = (F \cap M_1) + Z$ for any submodule $Z$ of $M$. Note that $f(F) = \pi_1 f (F \cap M_1) + \pi_2 f (F \cap M_2)$, because, $F = (F \cap M_1) \oplus (F \cap M_2)$ implies $f(F) = f(F \cap M_1) \oplus f(F \cap M_2) = \pi_1 f (F \cap M_1) + \pi_2 f (F \cap M_2)$. Let $M = N_1 \oplus N_2 + Z$. Then

$$M_1 = N_1 + (M_1 \cap (N_2 + Z)) = \pi_1 f j_1 (F \cap M_1) + (M_1 \cap (N_2 + Z)) = M_1 \cap [\pi_1 f j_1 (F \cap M_1) + (N_2 + Z)].$$

Thus $M_1 \leq \pi_1 f j_1 (F \cap M_1) + (N_2 + Z)$. Then $M_1 \leq \pi_1 f (F \cap M_1) + Z$, because, let $m_1 = \pi_1 f (y) + n_2 + z_0$, where $m_1 \in M_1, y \in F \cap M_1, n_2 \in N_2$ and $z_0 \in Z$. Since $Z = (Z \cap M_1) \oplus (Z \cap M_2), z_0 = z_1 + z_2$ where $z_1 \in Z \cap M_1$ and $z_2 \in Z \cap M_2$. Then $m_1 = \pi_1 f (y) + z_1$ and so $M_1 \leq \pi_1 f (F \cap M_1) + Z$. Similarly, $M_2 \leq \pi_2 f (F \cap M_2) + Z$. Thus $M = \pi_1 f (F \cap M_1) + \pi_2 f (F \cap M_2) + Z$. Therefore $M = f(F) + Z$.

Conversely, assume that $M = f(F) + Z$. Then $M = \pi_1 f (F \cap M_1) + \pi_2 f (F \cap M_2) + Z$. By modularity,

$$M_1 = \pi_1 f (F \cap M_1) + [M_1 \cap (\pi_2 f (F \cap M_2) + Z)]$$
and so
\[ M_1 = N_1 + [M_1 \cap (\pi_2 f(F \cap M_2) + Z)] = M_1 \cap (N_1 + \pi_2 f(F \cap M_2) + Z). \]

Thus \( M_1 \leq N_1 + \pi_2 f(F \cap M_2) + Z \). Therefore \( M_1 \leq N_1 + Z \). Similarly, \( M_2 \leq N_2 + Z \). Hence \( M = (N_1 \oplus N_2) + Z \). \( \Box \)

**Proposition 3.9.** Let \( F \) be a fully invariant submodule of a module \( M \). Assume \( \phi(F) \) has a supplement that is a direct summand of \( M \) for every \( \phi \in \text{End}_R(M) \) such that whenever \( M = M_1 \oplus M_2 \) then \( M_1 \) and \( M_2 \) are relatively projective. Then \( M \) is an \( I_F-H \)-supplemented module.

**Proof.** Let \( \phi \in \text{End}_R(M) \). By hypothesis, there exists a decomposition \( M = M_1 \oplus M_2 \) such that \( M = \phi(F) + M_2 \) and \( \phi(F) \cap M_2 \ll M_2 \) for some submodules \( M_1 \) and \( M_2 \) of \( M \). Since \( M_1 \) is \( M_2 \)-projective, by [9, Lemma 4.47], we get \( M = N \oplus M_2 \) for some submodule \( N \) of \( M \) such that \( N \leq \phi(F) \). Then \( \phi(F) = N \oplus (M_2 \cap \phi(F)) \). Let \( X \leq M \) with \( M = \phi(F) + X \). Then \( M = N + (M_2 \cap \phi(F)) + X \). As \( M_2 \cap \phi(F) \ll M_2 \), \( M = N + X \). Therefore \( M = N + X \) if and only if \( M = \phi(F) + X \). Hence \( M \) is \( I_F-H \)-supplemented. \( \Box \)

**References**


