Una mirada a medio siglo de cimentaciones laminares, métodos de cálculo e investigaciones asociadas en Cuba

A look at half a century of shells foundations, methods of calculation and associate research in Cuba

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Resumen

El trabajo presenta un resumen de las cimentaciones laminares más significativas construidas en Cuba en los últimos decenios y los desarrollos asociados a estas en cuanto a métodos de cálculo de láminas y cascarones de geometría compleja mediante superficies y cuerpos de referencia, a partir de la generalización del método de las solicitaciones proyectadas (Pücher, 1934) con el empleo de superficies de referencia (Hernández, 1970) y otros desarrollos alcanzados en la mecánica del sólido deformable mediante el Método de la Dualidad (Rianitsiyn, 1974; Castañeda, 1993) y la Analogía estático-geométrica en la mecánica del sólido deformable (Castañeda, 1985). Incluye además una síntesis de las investigaciones desarrolladas en los últimos años sobre el estado tenso-deformacional del suelo bajo cimentaciones laminares para chimeneas de 74,5 m en Centrales Azucareros (Cobelo, 2004), estudios comparativos realizados a estas con el empleo del MEF (González, 2010) y otros proyectos investigativos en ejecución actualmente (Álvarez, 2010).

Palabras Clave: Cimentaciones laminares, métodos de cálculo, coordenadas relativas, solicitaciones proyectadas, deformaciones proyectadas

Abstract

The work shows a summary of the most significant shells foundations built in Cuba in the last decades and the developments related to these in term of methods of calculation of plates and shells of complex geometry using reference surfaces and reference bodies, from the generalization of the projected solicitations method (Pücher, 1934) with the use of referential surfaces (Hernández, 1970) and other developments made in the mechanics of deformable solid by the Method of Duality (Rianitsiyn, 1974; Castañeda, 1993) and the Static-Geometric Analogy in the mechanics of the deformable solid (Castañeda, 1985). It also includes a summary of the research developed in recent years on the stress-strain states of soil under and inside shell foundations for chimneys of 74.5 m in sugar industries (Cobelo, 2004), comparative studies made of these with the use of the FEM (González, 2010) and other research projects currently running (Álvarez, 2010).

Keywords: Shell foundations, methods of calculation, relative coordinates, projected strains, projected deformations

1. Introducción

1. Introduction

Structural engineering in Cuba during the second half of the XX century and first decade of the XXI contains experiences and contributions to the calculation of shells that can not be described without looking at three very intertwined facts: the design and construction of shell foundations in buildings and elevated tanks (need); Relative Coordinates method generalized the method of “projected solicitations” on a Cartesian plane for the calculation of plates and shells² to any surface or body of reference (the solution);

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² El término “cáscara” o “cascarón” se utiliza en el texto para referirse a estructuras laminares (shells) de espesores medianos y gruesos, dentro de la definición brindada por la IASS (Asociación Internacional de Estructuras Laminares) cuyo comportamiento estructural exige considerar los efectos de fuerzas cortantes y momentos flectores sobre su superficie media, e incluso la posible composición de materiales tipo “sándwich” o de múltiples capas en general que requieran un estudio de distribución de tensiones, solicitaciones o corrimientos en su espesor

and research on stress-strain behavior of soils and foundation capacity under revolution shell foundations in short for chimneys with different numerical models, based on the Finite Element Method (applied research). The paper shows a summary of the most significant results of these facts, reveals links and refers to sources for its study and extension.

2. Discussion and development

The work is divided into three essential aspects of shell structures in Cuba within the period (1955-2010): design and construction of shell foundations buildings (Ruiz, 1962) and elevated tanks (Hernández et al, 1968); the rise of relative coordinates method (Hernandez, 1970) and its subsequent development as a generalization of the method of "projected solicitations" (Pucher, 1934), which marked the singularity in the calculation methods of shell; and research associated with the stress-strain state of the soil at the hearth and its supportive capacity on frictional soils ($C = 0$) under revolution shell foundation for short chimneys (74.5 m). It reveals the identity, difference, links, and dynamics of these aspects, which respond achievements, dissatisfactions and new research tasks that are undertaken.

2. 1 Design and construction of shell foundations in Cuba

Shell foundations have met in Cuba over fifty years. Shell foundation, as a structural solution for buildings, chimneys and elevated tanks, enrich the constructive heritage of the country. However, they lack the generalization that their economic, construction and design conditions grant due to the predominance of geotechnical design for stability, given the short duration of the dominant environmental burden (extreme wind) and the lack of significant seismic history.

The use of shell foundation drove the methods of calculation sheets and shells of complex geometry using reference surfaces and bodies in a line of generalization of the method of Pucher, and stimulated research on the behavior of layered soils under foundations with engineering applications.

Thus, in last century fifties it was designed and built in La Havana the first shell foundation as a polygonal shaped raft (folded plate) with braced upper slab to support a 24-storey building (94m high) capable of supporting winds with a lateral pressure of 300 kg/m² on a floor with a supportive capacity not exceeding 3 kg/cm², which has successfully overcome the test of time. This foundation, with a floor area of 2100 m² (77.2 x 27.2) and transverse bending modulus of 8m and 45cm thick (Figure 1), work of Dr. Ing Sixto Ruiz Alejo (1932), represented a new foundation for the epoch conception, with a 30% decrease in the cost of construction compared to a solution of raft ribbed trapezoidal beam at fixed prices at the time, not including the benefit of the tanks. (Ruiz, 1962)

Figura 1. Cimentación laminar de placa poligonal (Folded Plate) para edificio alto en Ciudad de la Habana (Cortesía del Dr. Ing. Sixto Ruiz Alejo)

Figure 1. Laminar foundation of folded plate for a high building in Habana City (compliments by Eng. PhD. Sixto Ruiz Alejo)

In addition, the limited availability and high cost of timber deposit INTZE type in the country, promoted a new type of elevated tanks for slide mold technology, called type “Güines”, in recognition to the first deposit, designed and build in a shell way, at Catalina de Güines, Habana Province in 1965 (Hernández et al, 1968).

This reservoir of 280m³ capacity and 27m high, was formed by a truncated cone shell foundation of reinforced concrete with a maximum diameter of 10mm minimum diameter and 2.20 in the shaft, 30cm average thickness and a basis angle of inclination of 30. The foundation was concrete against natural terrain, coupled with a rotating formwork, and closed at its top and bottom borders by a 70cm thick square ring. It also has a small circular plate 60 cm thick in the interior of the shaft. The shaft is a cylinder of 2.50 m outside diameter and 27m in height, constructed by sliding formwork (Figure 2). The cup and lid were concreted around the shaft before lifting. The new type required a system of metal formwork for concrete guidelines conical deposits and their covers around the shaft, until further lifting.

Figura 2. Primer depósito elevado tipo "Güira" con cemento laminar tronco-cónico, construido eb Catalina de Güines, la Habana, 1965 (Hernández y Rubiera, 1968)

Figure 2. First elevated water tank, type “Guira” with laminar trunk-conical foundation, built in Catalina de Guíñez, La Habana, 1965 (Hernandez y Rubiera, 1968)

The international development of "tower" type structures from the sixties of XXth century on produced a significant impetus to the shells foundations worldwide, and Cuba was no exception. Hermann Rühle, Vice President of the International Association of Shells Structures (IASS) pointed out, at that time: "The foundation best suited for slender structures radius between 0.1 and 0.05 is a closed conical shell with an outer ring. For this solution chimneys can have the disadvantage of requiring a considerable diameter ... In the case of poor soils and subsoils foundations circular tower-like structures require large diameter rings. But the shells are the most appropriate way to link the shaft and the outer ring "(Rühle, 1967).

In the late sixties, and in accordance with this idea, Dr. Ing. José (Pimpo) Hernández (1936-2003) designed and Architect José Licea Delgado (1930-1985) build in Cuba a revolutionary shells foundation with the Gaussian shape, following Havelka's proposal for tower type foundations (Leonhardt, 1967), with a maximum diameter of 10.80 m, 2.10 m deep, 20 cm thick and a shaft of the lift tower with 3.00 m in diameter (Figure 3) for a high tank capacity of 200 m³ and 20m height at the city of Matanzas, which has endured extreme winds of 240 km / h, with gusts to 320 km / h, measured at the site, structurally unaffected.

The requirements of this foundation, together with the background of the design and construction of a reservoir as a "water drop" shape for the treatment plant of the city of Santiago de Cuba a few years earlier (Figure 3), led Pimpo Hernández to develop, during 1969, a novel method of calculation sheets that generalized the method of "projects solicitations" on a Cartesian plane (Pücher, 1934) as a method called "Relative Coordinates", given the possibility of using other reference surfaces and select the most suitable for the calculation of each sheet, including the possibility that the equation of the plate can be replaced only by a scalar function of the lines of curvature of the reference surface in the normal vector direction (Hernández, 1970). The calculation of shells foundations of revolution with guideline of Gaussian form would not have been possible, at such moment, without a development of calculation methods such as the one which implied this generalization of the method of Pucher using Relative Coordinates, finally calculated over a polar plane.

Figura 3. Cimiento laminar en forma de Campana de Gauss (Matanzas) y depósito elevado en forma de "gota de agua" (Santiago de Cuba) del Ing. José E. (Pimpo) Hernández

Figure 3. Laminar foundation in the shape of a Gaussian Bell (Matanzas) and elevated water tank in the shape of a "water drop" (Santiago de Cuba) by Eng. Jose E. (Pimpo) Hernandez

The International Conference on "Foundations for tower-like structures" developed by IASS in 1970 and the VIII International Conference on Soil Mechanics and Foundations, 1973, marked a prolific synthesis of research on this topic. However, experimental studies and numerical models for problems related to soil-blade, the distribution of contact pressure between soil and foundation, the calculation of seat-supporting capacity of soils and the study of zones of contained plasticity by stress concentration effect on building shells on sand, soft clays and soft soils in general were still in its beginnings and identified as a problem of future research.

Whether it was 45cm thick in the "folded plate" of the building in La Habana, the 30 cm of foundation in Guines, or 20cm of the foundation as a bell curve in Matanzas, the shells foundation began its presence in structural engineering in Cuba in unison with its international environment and dragged it the development of calculation methods and concerns about the behavior of the soil, at a time when Nabor Carrillo, former Rector of the UNAM, prefaced the first books published in Mexico on "Soil Mechanics", written by Eulalio Juarez-Badillo and Alfonso Rico.

2.2 The development of the method of relative coordinates

The method of Relative Coordinates for the calculation sheet membrane was developed and applied to cases of roofs, tanks and foundations in a number of articles of the "Civil Engineering" Magazine at the beginning of the decade of the seventies (Hernández et al, 1973, 1974-a, 1974-b, 1975-a, 1975-b) from two basic ideas (Figure 4-a).

- To define an arbitrary reference surface S of known geometry, according to its lines of curvature (α_1, α_2) , by the position vector of an arbitrary point P on its surface as: $\vec{r} = \vec{r}(\alpha_1, \alpha_2)$
- To define the role $f = f(\alpha_1, \alpha_2)$ as relative scalar equation of sheet S* to be calculated with respect to arbitrary reference surface S and its lines of curvature (α_1, α_2) , through a distance PP* function, measured in the direction of surface of reference normal vector S, $(\vec{e}_n)$ until reaching each point S? of the real sheet to be calculated.

Whereupon, the equation for the surface S* of the real sheet is expressed in terms of lines of curvature of the reference surface as:

$$\vec{r}^*(\alpha_1, \alpha_2) = \vec{r}(\alpha_1, \alpha_2) + \frac{\vec{e}_n f}{2.1}(\alpha_1, \alpha_2) \quad (1)$$

And all the geometry of the real sheet S* can be expressed as a function of the Lamé parameters $[(A_1, A_2)]$ by the unit vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_n)$ and its derivatives at the reference surface S, and the relative equation $f = f(\alpha_1, \alpha_2)$ between the two surfaces, with which it can be generalize the concept of "designed solicitations" to relate the arc lengths and unit vectors on both surfaces by relations of the type:

$$\begin{aligned} K_i &= \left[\left(1 + \frac{f}{R_i} \right)^2 + \frac{f^2}{A_i^2 d\alpha_i^2} \right]^{1/2}; |\vec{dr}_i^*| = K_i A_i d\alpha_i = K_i ds_i \\ B_i &= \tan \beta_i = \frac{\frac{f}{A_i d\alpha_i}}{1 + \frac{f}{R_i}}; \quad \cos \varpi = \sin \beta_i \sin \beta_j; \quad D = \frac{\cos \beta_i \cos \beta_j}{\sin \varpi} \\ C_{ii} &= \frac{1 + B_i^2}{R_i} - \frac{B_i}{\rho_i} - \frac{B_i}{A_i d\alpha_i}; \quad C_{ij} = \frac{B_i B_j}{R_i} + \frac{B_i}{\rho_i} - \frac{B_j}{A_i d\alpha_j} \\ \vec{e}_i^* &= \vec{e}_i \cos \beta_i + \vec{e}_n \sin \beta_i; \quad \vec{e}_n^* = (-B_1 \vec{e}_1 - B_2 \vec{e}_2 + \vec{e}_n) D; \text{ para todo } (i; j = \overline{1;2}) \end{aligned} \quad (2)$$

$\alpha_i \rightarrow$ Coordenadas Gaussianas/Gaussian Coordinates; $A_i \rightarrow$ Parámetro de Lamé/Lamé Parameter; $ds_i \rightarrow$ Diferencial de arco/Differential arc
 $R_i \rightarrow$ Radio de Curvatura Gaussiana/Gaussian Curvature Radius; ρ_i Radio de Curvatura Geodésico/Geodesic Curvature Radius

Where $f=0$ it means that the reference surface S coincides with the average size of the sheet S^* to be calculated in lines of curvature, which causes that for that $K_i=1$, $B_i=\beta_i=0$ matter would $\alpha_1=x$; $\alpha_2=y$; y $f(\alpha_1, \alpha_2)=z(x,y)$ coincide with Pucher, and the reference surface would be a Cartesian plane (x,y) .

Figura 4. Coordenadas Relativas entre dos superficies (4 a) y entre dos medios tridimensionales (4 b)
Figure 4. Relative coordinates between two surfaces (4a) and between two tridimensional means (4b)

The method of Relative Coordinates opened at least three new ways to calculate shells structures:

- It established the geometric base for generalizing Pücher's ideas in relation to the introduction of "projected solicitations" on a Cartesian plane, creating the possibility of selecting the surface of reference S more suitable for the calculation of each shell according to its shape, charge and/or conditions of support (Example: polar plane or a cylinder coaxial for a shell of revolution with loads in the direction of the vector $\vec{e}_n$ o $\vec{e}_n^*$; Cartesian plane for the calculation of a rectangular base hyperbolic paraboloid, parallel to its asymptotes, etc.)

- It provided a new way to calculate shells of complex shape without analytical expression to describe their average size, setting, through analytical or numerical way, an equation concerning the distances between the medial surface of the shell S^* and an area of reference known as S by a function type $f=f(\alpha_1, \alpha_2)$
- It extended the possibilities of calculating in "projected solicitations", boundary problems having a simpler geometric shape on a reference surface S than on the actual average surface S^* of the shell, as it is the case of two cylinders of equal diameter, that cut their long axes at 90 degrees, and whose curve of intersection in warped space, projected as a contour, a straight diagonal line over a Cartesian plane, selecting S as a reference surface for its calculation.

A test of the potential of this geometric approach arose almost simultaneously in the second half of the seventies when a group of researchers from the Kazan Aviation Institute (Paimuchin et al., 1975, 1977, 1980), without using Pücher's method of "projected solicitations", proposed a similar geometric equation for the approximate calculation of shells lowered and set the basic equations for calculating shells of complex shapes such as "flashlights" and other aircraft parts from this geometric perspective, with linear and nonlinear approaches for the calculation of shells of "sandwich" type of constant and variable thickness by calculating approximate surface.

While in Cuba, since the mid-seventies until the nineties, continued research on the method of relative coordinates with "projected solicitations" and new results were achieved that allowed the planning of the relative coordinates with traditional methods of differential geometry of shells not referred to lines of curvature (Castañeda, 1984-a) and its generalization for thick and media shells, which can not be reduced to an average size (Castañeda, 1985, 1995), through the introduction of the equation between two three-dimensional media S_z and S_z^* , and developments resulting in differential geometry (Figure 4-b),

$$\overrightarrow{r_z^*}(\alpha_1, \alpha_2, z^*) = \overrightarrow{r}(\alpha_1, \alpha_2) + \overrightarrow{e_n} f(\alpha_1, \alpha_2) + z^* \overrightarrow{e_n^*} \quad (3)$$

Where the geometry of the real three-dimensional media, $S_z^*(\alpha_1, \alpha_2, z^*)$, is based on the geometry of the referential three-dimensional S_z^* (α_1, α_2, z) media for everything: $z = z^* D$; o $z = z^*/D$ providing that:

$$\begin{aligned} K_{iz} &= \left[\left(1 + \frac{f+z}{R_i} - \frac{zB_j}{\rho_i} - \frac{\partial(zB_i)}{A_i \partial \alpha_i} \right)^2 + \left(\frac{zB_i}{\rho_i} - \frac{\partial(zB_j)}{A_i \partial \alpha_i} \right)^2 + \left(\frac{\partial f}{A_i \partial \alpha_i} + \frac{\partial z}{A_i \partial \alpha_i} + \frac{zB_i}{R_i} \right)^2 \right]^{\frac{1}{2}} \\ K_{nz} &= \left[\left(\frac{\partial(B_1 z)}{A_n \partial \alpha_n} \right)^2 + \left(\frac{\partial(B_2 z)}{A_n \partial \alpha_n} \right)^2 + \left(\frac{\partial z}{A_n \partial \alpha_n} \right)^2 \right]^{\frac{1}{2}}; \quad \left| \overrightarrow{dr_{iz}^*} \right| = K_{iz} A_i d\alpha_i = K_{iz} ds_i \\ G_i &= \tan \gamma_i = \frac{\frac{zD_{ii}}{K_i \cos \beta_i}}{\left(1 + \frac{z \cos \beta_i C_{ii}}{K_i} \right)}; \quad D_{ii} = D [C_{ij} - B_i B_j \cos^2 \beta_i C_{ii}] \\ \overrightarrow{e_{iz}^*} &= \overrightarrow{e_i^*} \cos \gamma_i + \overrightarrow{e_j^*} \sin \gamma_i; \quad \overrightarrow{e_j^*} = \frac{1}{\sin \omega} \overrightarrow{e_i^*} - \frac{\cos \omega}{\sin \omega} \overrightarrow{e_j^*}; \quad \text{para todo } (i; j = \overline{1; 2}) \end{aligned} \quad (4)$$

In these works (Castañeda, 1985, 1995) were established the conditions of orthogonality of the unit vectors in the average surface of the shell by its equation of relative coordinates $\beta_1 = 0$, o $\beta_2 = 0$, associated to ($\varpi = 90^\circ$); the case where the conditions for lowering a shell on its surface when not required solicitations and projected deformations (si β_1 y β_2 if and tend to zero simultaneously), V.N. Paimuchin correcting criteria on the conditions of orthogonality and of debasement of these shells in relation to their surface calculation and establish geometric relations in relative coordinates for an orthogonal reference system, in correspondence with the geometric criteria for a point of a thick shell (Goldenveizer, 1953).

Thus, membrane theory included the "projected displacements" (Castañeda, 1981-a) and "projected deformations" (Castañeda, 1982-a, 1982-b, 1982-c) to move the whole process of calculation to the reference surface on hiperstatic problems, supported by the method of dual static-geometric equations (Rianitsyn, 1974) and the method of virtual work (Hernández, 1982), creating conditions for obtaining the basic equations of the theory of flexion of Love-Kirchhoff type and Timoshenko type in Relative Coordinates to solicitations, deformations and displacements projected on the reference surface (Castañeda, 1983-a, 1983-by 1983-c),

which was vital to provide integration and complementation to the method where "no membrane" extend the results to structural laboratory experiments in physics (1984-b), and form a static-geometric view point that generalizes Pücher's method for any reference surface membrane theory, bending and analysis of thick shells, which was the most significant result of this stage in the development of methods of calculation (Figure 5).

The equilibrium equations (2.5), geometric (2.6), physical or constitutive (2.7), of projected solicitations to real solicitations (2.8) and deformation projected to actual deformation (2.9) in Relative Coordinates to the theory of bending type Love-Kirchoff in which the straight and normal element to the shell middle surface remains straight and normal after deformation, has for ($f=0$) the classic cases of Gauss intrinsic coordinates, and for $\alpha_1=x$; $\alpha_2=y$; $\gamma f(\alpha_1, \alpha_2)=z(x, y)$ Pücher's equation showing the validity of this approach and the new possibilities created to establish and solve boundary problems (polar plane, cylindrical, coaxial, etc..) only by introducing the geometric parameters of the first and second quadratic form if it has a function of the type $f=f(\alpha_1, \alpha_2)$.

Furthermore, the method of Duality (Table 1), applied between the equilibrium equations (2.5) and the geometrical equations (2.6), confirmed the relationship between the projected displacement vector (linear and angular) as a result of neglecting deformations shear (Q_i) in Love-Kirchhoff Theory and consider the infinite stiffness of the shell with respect to the torsion of the normal axis to its average surface $\vec{e}_n^*$, as it corresponds to this approach.

Figura 5. Modelación del Método de Pücher en Coordenadas Relativas para los diferentes niveles de análisis de cáscaras: a) Membrana; b) Flexión; c) Cáscaras gruesas.

Figure 5. Modeling of Pucher Method in Relative Coordinates for different levels of shells analysis a) membrane; b) flexion; c) thick shells

Equilibrium equations in projected solicitations

$$\begin{aligned}
& \left(\frac{1}{\rho_j} + \frac{B_i}{R_i} \right) N_i + \left(\frac{1}{\rho_i} + \frac{B_j}{R_j} \right) N_{ij} - \frac{N_j}{\rho_j} + \frac{N_{ji}}{\rho_i} + \frac{Q_i}{R_i} + \frac{\partial N_i}{A_i \partial \alpha_i} + \frac{\partial N_{ji}}{A_j \partial \alpha_j} + q_i = 0 \\
& C_{ii} N_i + C_{ij} N_{ij} + C_{ji} N_{ji} + C_{jj} N_j + \left(\frac{B_i}{R_i} - \frac{1}{\rho_i} \right) Q_i + \left(\frac{B_j}{R_j} - \frac{1}{\rho_j} \right) Q_j - \frac{\partial Q_i}{A_i \partial \alpha_i} - \frac{\partial Q_j}{A_j \partial \alpha_j} - q_n + B_i q_i + B_j q_j = 0 \\
& \frac{\partial M_{ij}}{A_i \partial \alpha_i} + M_{ij} \left(\frac{1}{\rho_j} + \frac{B_i}{R_i} \right) + \frac{\partial M_j}{A_j \partial \alpha_j} + \frac{M_j}{\rho_i} - M_i \left(\frac{1}{\rho_i} + \frac{B_j}{R_j} \right) + \frac{M_{ji}}{\rho_j} + N_{ij} \frac{\partial f}{A_i \partial \alpha_i} + \\
& - N_{ji} \left(1 + \frac{f}{R_i} \right) B_i - \left(1 + \frac{f}{R_j} \right) Q_j + m_i = 0; \quad \text{para todo } (i, j = \overline{1, 2}) \\
& - C_{ij} M_i + C_{ji} M_j + C_{ii} M_{ij} - C_{jj} M_{ji} - Q_i \left(1 + \frac{f}{R_i} \right) B_j - Q_j \left(1 + \frac{f}{R_j} \right) B_i - m_j B_j + m_i B_i + \\
& + N_{ij} \left[B_i \frac{\partial f}{A_i \partial \alpha_i} - \left(1 + \frac{f}{R_i} \right) (1 + B_j^2) \right] - N_{ji} \left[\left(1 + \frac{f}{R_j} \right) (1 + B_i^2) - B_j \frac{\partial f}{A_j \partial \alpha_j} \right] = 0
\end{aligned} \tag{5}$$

Where:

$q_1; q_2; q_n; m_1; m_2$: Forces and moments are mass flow in the direction $\vec{e}_1, \vec{e}_2, \text{ y } \vec{e}_n$

$Q_1 \text{ y } Q_2$: Are the projected shear

Geometric equations projected deformations and displacements

$$\begin{aligned}
\varepsilon_i &= \delta_{ii} + B_i \delta_{i3}; & \gamma_{ij} &= K_1 \cos \beta_1 (\delta_{21} + B_1 \delta_{23}) + K_2 \cos \beta_2 (\delta_{12} + B_2 \delta_{13}) \\
\kappa_i &= -\delta_{ij}'; & \kappa_{ij} &= -\delta_{ii}';
\end{aligned}$$

$$\frac{\partial \vec{u}}{A_i \partial \alpha_i} = \delta_{ii} \vec{e}_i + \delta_{ij} \vec{e}_j + \delta_{i3} \vec{e}_n; \quad \frac{\partial \vec{\Omega}}{A_i \partial \alpha_i} = \delta_{ii}' \vec{e}_i + \delta_{ij}' \vec{e}_j + \delta_{i3}' \vec{e}_n; \quad \frac{1}{2} \text{rot} \vec{u} = \vec{\Omega}$$

$$\begin{aligned}
\delta_{ii} &= \frac{\partial u_i}{A_i \partial \alpha_i} + \frac{u_j}{\rho_i} + \frac{u_n}{R_i}; & \delta_{ii}' &= (-1)^i \left[\frac{\partial \psi_j}{A_i \partial \alpha_i} - \frac{\psi_i}{R_i} \right] + \frac{\delta}{R_i} \\
\delta_{ij} &= \frac{\partial u_j}{A_i \partial \alpha_i} - \frac{u_i}{\rho_i}; & \delta_{ij}' &= (-1)^i \left[-\frac{\partial \psi_i}{A_i \partial \alpha_i} - \frac{\psi_j}{\rho_i} \right] \\
\delta_{i3} &= \frac{\partial u_n}{A_i \partial \alpha_i} - \frac{u_i}{R_i}; & \delta_{i3}' &= \frac{\partial \delta}{A_i \partial \alpha_i} + (-1)^i \left(-\frac{\psi_j}{R_i} \right)
\end{aligned} \tag{6}$$

Constitutive physical equations and deformations projected solicitations.

$$\begin{aligned} N_i &= \frac{2Eh}{1-\mu} \left[\frac{K_j \cos^2 \beta_i}{K_i \sin^3 \omega} \varepsilon_i + \frac{D}{\sin^2 \omega} \varepsilon_j - \frac{\cos \omega \cos \beta_i}{K_i \sin^3 \omega} \gamma_{ij} \right] \\ N_{ij} &= \frac{2Eh}{1-\mu^2} \left[\frac{K_j \cos \beta_j}{K_i \sin^3 \omega} \left[-\frac{\cos \omega \cos \beta_i}{K_i \sin^3 \omega} \varepsilon_i - \frac{\cos \omega \cos \beta_j}{K_j \sin^3 \omega} \varepsilon_j + \frac{\gamma_{12}(1 + \cos^2 \omega - \mu \sin^2 \omega)}{2K_i K_j \sin^3 \omega} \right] \right. \\ M_i &= \frac{2Eh^3}{3(1-\mu^2) \sin \omega} \left(\frac{K_j \cos^2 \beta_j}{K_i} \kappa_i + \mu \cos \beta_i \cos \beta_j \kappa_j \right) \\ M_{ij} &= \frac{2Eh^3 \kappa_{ij} K_j \cos \beta_i}{3(1+\mu) \sin \omega}; \text{ Siempre que : } \left(1 + \frac{Z^*}{R_i^*} \right) \approx 1 \text{ para todo } (i; j = \overline{1; 2}) \quad (7) \end{aligned}$$

E: Módulo de elasticidad/Elasticity modulus; **μ:** Coeficiente de Poisson/Poisson ratio; **h:** Espesor/Thickness
N_i, N_j: Solicitación normal proyectada/Projected normal solicitation; **N_{ij}:** Solicitación tangencial proyectada/Projected tangential solicitation
M_i, M_j: Solicitación de torsión proyectada/Projected torsional solicitation; **M_{ij}:** Solicitación de flexión proyectada/Projected bending solicitation
ε_i, ε_j: Deformación longitudinal proyectada/Projected longitudinal deformation; **γ_{ij}:** Distorsión proyectada/Distortion projected
κ_i, κ_j: Deformación de torsión proyectada/Projected torsional deformation; **κ_{ij}:** Deformación de flexión proyectada/bending deformation projected

Relationship among projected solicitations and actual solicitations

$$\begin{aligned} N_i &= N_i^* K_j \cos \beta_i - DB_i Q_i^* K_j; & N_{ij} &= N_{ij}^* K_j \cos \beta_j - DB_j Q_i^* K_j; & Q_i &= \frac{K_j Q_i^*}{D^2} \quad (8) \\ M_{ij} &= M_{ij}^* K_j \cos \beta_i; & M_i &= M_i^* K_j \cos \beta_j; \end{aligned}$$

Relationship among projected deformations and actual deformations

$$\begin{aligned} \varepsilon_i^* &= \frac{\cos \beta_i}{K_i} \varepsilon_i; & \gamma_{12}^* &= \frac{\gamma_{12}}{K_1 K_2 \sin \omega} - \left(\frac{\cos \beta_1}{K_1} \varepsilon_1 + \frac{\cos \beta_2}{K_2} \varepsilon_2 \right) \frac{\cos \omega}{\sin \omega} \quad (9) \\ \kappa_i^* &= \frac{\cos \beta_j}{K_i} \kappa_i; & \kappa_{ij}^* &= \frac{\cos \beta_i}{K_i} \kappa_{ij}; \end{aligned}$$

The method of Relative Coordinates seeks new developments today pointing to the obtention of Compatibility Equations of Deformity type Saint Vennat and Airy stress function, Maxwell or Morera through Duality method (Rianitsyn, 1974) and static -geometric analogy (Castañeda, 1985, 1993) for the solicitations and deformations calculation projected by using semi-inverse and inverse methods associated with the hipereleasticity degree of the shells, with Finite Difference solutions, where the operator transpose of the ordinary differential equation:

$$P(x) = L[N(x)] = a^{(0)}N + a^{(1)}\frac{d(b^{(1)}N)}{dx} + \dots + a^{(k)}\frac{d^{(k)}(b^{(k)}N)}{dx^k}; \text{ tiene la forma}$$

and the transposed operator of differential equation in partial derivatives:

$$\varepsilon(x) = L^*[U(x)] = a^{(0)}U - b^{(1)}\frac{d(a^{(1)}U)}{dx} + \dots + (-1)b^{(k)}\frac{d^{(k)}(b^{(k)}U)}{dx^k} \quad (10)$$

$$P(x) = \sum_k \sum_l \sum_m \dots a^{(k,l,m)} \frac{\partial^k \partial^l \partial^m \dots (b^{(k,l,m)}N)}{\partial x^k \partial y^l \partial z^m}; \text{ tiene la forma:}$$

$$\varepsilon(x) = \sum_k \sum_l \sum_m \dots (-1)^{(k+l+m)} a^{(k,l,m)} \frac{\partial^k \partial^l \partial^m \dots (b^{(k,l,m)}U)}{\partial x^k \partial y^l \partial z^m} \quad (11)$$

When $a^{(0)}$, $a^{(1)} \dots a^{(m)}$, and $b^{(0)}$, $a^{(1)} \dots b^{(m)}$, are functions of the different variables and ($\in \mathbb{N}$), which allows to obtain the geometric equations transposing equations of equilibrium (Rianitsyn, 1974) as shown at Table 1 introducing equilibrium equations per rows (2.5) and extracting per column geometric equations (2.6).

Tabla 1. Relación transpuesta del método de la dualidad para la teoría de Love-Kirchoff
Table 1. Relationship transpose duality method for the Love-Kirchoff theory

Solicitudes	N_1	N_2	N_{12}	N_{21}	Q_1	Q_2	M_1	M_2	M_{12}	M_{21}	M_{1N}	M_{2N}	u_1, ψ_1	N_i, M_i
GLE^*														
$\Sigma F(\bar{e}_1) = 0$	$A_1 \frac{\partial(\cdot)}{\partial a_1} + \dots$	...	$\frac{A_1 A_2}{\rho_1} + \dots$	...	$\frac{A_1 A_2}{R_1}$									u_1
$\Sigma F(\bar{e}_2) = 0$	$-\frac{A_1 A_2}{\rho_1}$	...	$A_2 \frac{\partial(\cdot)}{\partial a_1} + \dots$	...		$\frac{A_1 A_2}{R_2}$								u_2
$\Sigma F(\bar{e}_N) = 0$	$A_1 A_2 \frac{\partial(\cdot)}{\partial a_1} + \dots$	...	$A_2 A_1 \frac{\partial(\cdot)}{\partial a_1} + \dots$	...	$\frac{A_1 A_2}{\rho_2} + A_2 \frac{\partial(\cdot)}{\partial a_1}$	$\frac{A_1 A_2}{\rho_1} + A_1 \frac{\partial(\cdot)}{\partial a_2}$							ψ_N	
$\Sigma M(\bar{e}_1) = 0$						$-(1 + \frac{f}{R_2}) A_1 A_2$	$-\frac{A_1 A_2}{\rho_1}$	...	$A_2 \frac{\partial(\cdot)}{\partial a_1} + \dots$	...	$\frac{A_1 A_2}{R_1}$		ψ_1	
$\Sigma M(\bar{e}_2) = 0$					$-(1 + \frac{f}{R_1}) A_1 A_2$		$-A_2 \frac{\partial(\cdot)}{\partial a_2} + \dots$	...	$-\frac{A_1 A_2}{\rho_1}$	...		$\frac{A_1 A_2}{R_2}$	ψ_2	
$\Sigma M(\bar{e}_N) = 0$								...	$-\frac{A_1 A_2}{R_1}$	...	$\frac{A_1 A_2}{\rho_2} + A_1 \frac{\partial(\cdot)}{\partial a_1}$	$\frac{A_1 A_2}{\rho_1} + A_2 \frac{\partial(\cdot)}{\partial a_2}$	δ	
ε_i, κ_i	$\varepsilon_1 A_1 A_2$	$\varepsilon_2 A_1 A_2$	$\gamma_{12} A_1 A_2$	$\gamma_{21} A_1 A_2$	NO HAY	NO HAY	$\kappa_1 A_1 A_2$	$\kappa_2 A_1 A_2$	$\kappa_{12} A_1 A_2$	$\kappa_{21} A_1 A_2$	NO HAY	NO HAY	Deformaciones	GLG^*
$\Sigma F_i, \gamma_i M_i = 0$														

Legenda:
 $GLG^* \rightarrow$ Grados de libertad geométricos;
 $E \rightarrow$ Entrada;
 $GLE^* \rightarrow$ Grados de libertad estáticos
 $S \rightarrow$ Salida

2.3 Research of stress-strain state of soils under revolutionary shells foundation for Finite Element Method (MEF)

In the first decade of XXI century Cuba has done research on models based on Finite Element Method (SIGMA / W, PLAXIS,

ABAQUS) on the behavior revolution shells foundations (straight and parabolic guideline) (Figure 6) and foundation soils under laminar axial-symmetric load (Cobelo, 2004, Gonzalez, 2010) for the determination of the contact pressure distribution, seating, supportive capacity of soils and solicitations in the structure, considering the soil-blade interaction. This work was carried out in three stages of research as an alternative foundation for short stacks of 74.5 m in height, typical Cuban structure for sugar mills.

Figura 6. Características generales de la modelación de la cimentación por el MEF
Figure 6. General characteristics of foundation modeling by MEF

The studies for these fireplaces were done on a foundation geotechnically designed of 16m diameter composed of two layers: an inverted dome 8m at the bottom of shaft, which works entirely in compression, and a truncated cone plate embedded in the shaft and free at the outer edge of the foundation. The truncated cone sheet in all cases was divided into 10 cross sections 3, 5 and 9 points defined on each section with which were evaluated the handle radial (N_r) and circumferential (N_ϕ) in the middle surface. When structural concrete was given a behavioral model linear-elastic, with $E=2,16 \times 10^7 \text{ kPa}$ and Poisson Coeficient $\mu=0,176$.

In the first stage (Cobelo, 2004) it was developed an experiment 3² with permutations of soils in the core, below the floor slab; value of the angle of internal friction ϕ' ($\phi'_1=20^\circ$, $\phi'_2=30^\circ$ y $\phi'_3=40^\circ$) and Elasticity Moduls ($E_{01}=7260 \text{ kPa}$, $E_{02}=11000 \text{ kPa}$ y $E_{03}=30000 \text{ kPa}$), with three conical geometries, constructively competitive and differentiated by the angle (α) of the guideline inclination in reference to the base plane ($26,5^\circ$, 35° y 45°) and a relation f/a (0,5; 0,7 y 1,0), determining 27 numeric experiments,

done with GeoSlope v5.13 SIGMA/W Software, repeated for elastic and elasto-plastic models (Mohr-Coulomb Type) of soil behavior, up to a total of 54 studied cases.

In the second stage (González, 2010) another numerical experiment was raised where 45 cases were considered of homogeneous soils in the core, below the floor slab for the same values of friction angle ϕ' and Module elasticity of soils E_{0i} for the three previous conical trunk geometries and two new parabolic geometries of negative Gaussian curvature (the worst) with the use of Plaxis 2D v8.2, to confront them with the results of the first stage.

In the third stage, still running, new numerical experiments are performed for all permutations of soils in the core, below the floor slab level with models of elastic and elasto-plastic behavior, using ABAQUS program on sheets of revolution of double curvature of positive Gaussian curvature, increasing the number of nodes on each cross section of the sheet, which are compared with the results of inverse solutions in Relative Coordinates and Theory of Flexion Love-Kirchhoff type by the application of Finite Differences method.

In reference to radial solicitations (N_r) and circumferential (N_ϕ) in the average surface of the sheet (Figure 7) the results of the first two steps show that the influence of a type or another of frictional soil and its disposition in one or another of the identified zones has less significance, for the same geometry, than the elastic or elasto-plastic model of physical constitutive behavior assumed, and that such influence is even minor in the measure that the sheet augments its relation f/a . The first two graphs N_r y N_ϕ appearing at the top of Figure 7 for $f/a = 0,5$ ($26,5^\circ$) correspond to a linear-elastic behavior of the soil, the next two ones to an elasto-plastic behavior of Mohr-Coulomb type for the same geometry, and the two last ones to the same lineal-elastic model for $f/a = 1$ (45°). In each graph are shown the variations of N_r y N_ϕ for the studied 9 combinations of soils, confirming its characteristic behavior in these cases.

Figura 7. Gráficos de N_r y N_p para diferentes relaciones de f/a y modelos de suelos

Figure 7. Graphs for N_r and N_p for different f/a relations and soil models

The results obtained by computer modeling with application of MEF considering soil-structure interaction, allowed to enter "correction factors to the membrane solicitations" F_c^r and F_c^p and the Effect of Simple Alteration.

The vertical stress distributions (σ_y) at the sill, corresponded in all cases to Szechy physical tests (1965), Nicholls and Izadi (1968) and Kurian (1983), confirming that there is vertical stress concentration towards the layered conical edges of the foundation and a plastic redistribution beneficial for the central area (Figure 8).

Vertical displacements (s_y) on the foundation sill showed that the soil confined core is not incompressible and absorbs vertical deformation in part of the sill area, this being more pronounced toward the outer edge, where there are areas of contained plasticity near the end of the rolled edge ring (Figure 9).

Figura 8. Distribución de tensiones en la solera para suelo elástico y elasto-plástico en $f/a = 0,5$

Figure 8. Curb stress distributions for elastic soil and elastic-plastic soil in $f/a = 0.5$

Figura 9. Distribución de asentos (s_y) en la solera para suelo elástico y elasto-plástico en $f/a = 0,5$

Figure 9. Curb foundation distribution (s_y) for elastic soil and elastic-plastic soil in $f/a = 0.5$

Studies of q_u , load capacity of the soil under the shell foundation analyzed show that these values can be achieved up to 40% higher than load capacity Terzaghi's equation, 34% higher than those determined by Hansen's criteria, and up to 20% calculated on Meyerhof's criteria, depending on the angle of inclination of the guideline and soil types present in the nucleus, the filler and the base. Table 2 shows the q_u load capacity results obtained by FEM for geometric variation $f/a=0,7$ in comparison with Terzaghi, Hansen and Meyerhof analitic methods, depending on the soil friction angle.

Thus, and similar to other researchers (Rahman, 1987) was estimated a correction factor F_q that modifies the coefficient N_q calculated by Reissner (1924) [Bracha (2000), Cuban Standard for Geotechnical Design of Superficial Foundations (2007)] reconciles the values of the capacity of frustoconical foundations with those determined for displaced shell foundation of triangular section (Hanna y Rahman, 1990), with a new correction for the effect of soil-shell. In addition, S_s established form factors that added to the equation of Terzaghi load capacity (with load capacity coefficients corrected by the factor F_q), taking into account the influence of the shape of the foundation in the increase of q_u (Table 3).

Tabla 2. Resultados de capacidad de carga q_u en modelo de MEF y métodos analíticos en $f/a = 0,7$
Table 2. Results of loading capacity q_u in MEF model and analytical methods in $f/a = 0.7$

$\Phi(^{\circ})$	M.E.F $q_u(\text{kN/m}^2)$	Terzaghi $q_u(\text{kN/m}^2)$	% Dif.	Hansen $q_u(\text{kN/m}^2)$	% Dif.	Meyerhof $q_u(\text{kN/m}^2)$	% Dif.
20	1329.00	810.0	+39	946.80	+28	1058.0	20
25	2406.00	1589.3	+34	1882.60	+21	2245.8	7
30	5282	3229.0	+39	3805.10	+28	4884.7	8
35	12125	6928.0	+43	7967.80	+34	11375.6	6
40	26688	16108.0	+40	17656.60	+34	28860.7	-8

Tabla 3. Coeficientes de forma corregido S_s para la capacidad de carga bajo láminas tronco-cónica
Table 3. Amended shape coefficient S_s for loading capacity under trunk-conical plates

S_s	f/a		
$\phi(^{\circ})$	0.5	0.7	1
20	1.11	1.13	1.16
30	1.03	1.06	1.08
40	1.10	1.14	1.20

$$q_u = \left(\frac{1}{2} \gamma B N_{\gamma t} + q^* N_{qt} + C N_{ct} \right) S_s$$

In the second stage of the investigations it was used an "interface" with an elasto-plastic Mohr-Coulomb type behaviour, affected by a $R_{inter} = 0.666$ coefficient [Ibañez (2000) and Cobelo (2004)], over all the contact surface between soil and foundation plate. Based on the experiences of other researchers [Huat & Mohamed (2006) and Esmaili & Hataf (2008)] and the use of Plaxis 2D (Gonzalez, 2010) were modified and expanded the previous models, taking advantage of this tool: automatic generator and unstructured meshes and particular data output of the load-deformation relation in a virtual trial-load capacity through the calculation option "c- ϕ ' reduction" which established other graph criteria for the determination of ultimate load capacity. The results of calculation according to this criterion are presented graphically (Figure 10) for each depth of foundation D_f , and q_u curves are included according to Terzaghi and Hansen criteria [Braja (2000), Cuban Standard for Geotechnical Design of Superficial Foundations (2007)] for foundations of flat circular equal base area, as the one obtained in the first stage for the truncated cone sheet of straight guideline $f/a = 0.7$.

The research planned in the third stage, with the ABAQUS program and the use of semi-inverse methods in Relative Coordinates, Theory of Finite Differences and Flexion, plans to introduce the effect of the Horizontal Force and the eccentricity of the vertical load on computational modeling shell foundations of these short stacks.

Figura 10. Resultados de capacidad de carga q_u para diferentes D_f , según método "c- ϕ ' reduction"

Figure 10. Results of loading capacity q_u for different D_f , according to "c- ϕ ' reduction" method

3. Conclusions

The past half century shell foundations enriched the built heritage in Cuba and the engineering professional culture of the country, developing new methods of calculation and research partners, as part of an overall global reach process. Among the conclusions of this period there are three main aspects:

- The good structural behavior of shell foundations of elevated tanks built in Cuba in the early 60's of last century coincides with the beneficial results from the point of view of pressure distribution, settlements in the hearth and ability load level as more recent research show in tower-like structures (short chimneys) with geotechnical and computational models based on the application of the Finite Elements Method (FEM). The distributions of stresses and hearth settlements obtained by this method show that the soil contained within the shell is considerably deformable, as posed by the design assumptions of that time, because it absorbs some of these tensions and redistributes better stresses and strains with respect to a circular rigid base foundation.
- In the range of geometries analyzed (guidelines to $\alpha=26,5^\circ$ ($f/a=0,5$) and $\alpha=45^\circ$ ($f/a=1,0$)), constructively acceptable for this type of shells, the results on the deviations in the physical properties of soil constitutive frictional ($c = 0$), confirm that these variations do not introduce significant differences, for engineering purposes, on the value of the solicitations N_r y N_ϕ depending on the angle of internal friction of soil. Furthermore, these results show that small variations in the shell guideline in the construction technology used (in situ or precast) do not introduce large modifications in the internal solicitations of it, while the degree of compaction of soils within the shell and over the hearth level, which are variables that control the engineer during the execution of the work, may have greater significance in the stress distribution and settlement of the foundation level.

- The development of models and methods of calculation sheets and shells based on the method of Relative Coordinates, together with the potential of numerical methods and computer application (Finite Differences Method, FEM, etc..) shows each day less barriers for engineers to design and build competitive shell foundations, even considering more complex models of soils in its physical-mathematical representation, but which are much closer to reality and experimental evaluation of their specific differences (Mohr-Coulomb, Drucker-Prager, etc.).

4. Referencias / References

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