Modelación numérica del concreto simple con elementos finitos mediante la teoría de la plasticidad y la función de fluencia de Hu y Schnobrich

Numerical modeling of the simple concrete with finite elements by means of the plasticity theory and yielding function of Hu and Schnobrich

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Resumen

Este artículo describe la formulación, implementación y validación de un modelo constitutivo en el método de los elementos finitos, que represente el comportamiento mecánico del concreto simple sometido principalmente a compresión, considerando estado plano de esfuerzos y deformaciones infinitesimales. Este modelo se basa en la formulación general de la teoría de la plasticidad (Simó & Hughes 1998), considerando una regla de flujo no asociado, donde el potencial plástico está definido por el criterio de fallo de von Mises y la función de fluencia corresponde a aproximaciones empíricas realizadas por Hu y Schnobrich (1989). Se formuló un algoritmo implícito de integración numérica para resolver el problema no lineal dado por el modelo constitutivo del material. El modelo constitutivo presentado en este artículo se implementó en el programa de análisis no lineal con elementos finitos a código abierto HYPLAS (de Souza Neto et al. 2008) y el postproceso se realizó con el programa GiD (CIMNE 2008). Finalmente se presenta la comparación de la respuesta estructural de los paneles sometidos a fuerzas contenidas en su plano ensayados experimentalmente por Kupfer y otros (1969), contra la simulación numérica utilizando el modelo propuesto. También se presenta un ejemplo de la aplicación del modelo propuesto a la simulación de vigas sometidas a flexión.

Palabras Clave: Modelos constitutivos de plasticidad, mecánica computacional, método de los elementos finitos, concreto simple.

Abstract

This article describes the formulation, implementation and validation of a constitutive model in the finite element method, which it represents the mechanical behavior of plain concrete subjected mainly to compression, considering the plane stress state and infinitesimal strain. This model is based on the general formulation of the theory of plasticity (Simó & Hughes 1998), considering a non-associated flow rule, where the plastic potential is defined by the failure criterion and von Mises and yield function corresponds to empirical approaches developed by Hu and Schnobrich (1989). An implicit algorithm of numerical integration was formulated to solve the nonlinear problem given by the material constitutive model. The constitutive model presented in this paper was implemented in the nonlinear analysis finite element program with open source HYPLAS (de Souza Neto et al. 2008) and the post-process was performed using GiD (CIMNE 2008). Finally we present the comparison of the structural response of panels subjected to forces contained in its plane tested experimentally by Kupfer and others (1969), against the numerical simulation using the proposed model. Also an application example of the proposed model to the beams simulation subjected to bending was presented.

Keywords: Plasticity constitutive models, computational mechanics, finite elements method, simple concrete

1. Introduction

The classic plasticity theory was initially formulated for metals, which behavior is quite different from concrete's behavior. However, from a microscopic point of view, metals and concrete have similar characteristics, especially in the range previous to yielding, which lead us to represent such materials by means of the plasticity theory. For instance, concrete has a non-linear behavior during loading process and undergoes an irreversible strain after removing load and; it has a ductile behavior in the range of three-axial or biaxial compression (Chen & Han 2007).

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There is a wide variety of constitutive models based on the plasticity theory, which intends to represent materials mechanic behavior - particularly concrete - most of them establish a yielding function and a hardening or softening flow law. (Babu et al., 2005, De Souza Neto et al., 2008, Simó & Hughes 1998).

The yielding function, part of the plasticity model, may come up from classic yielding criteria, such as criteria developed by Tresca, von Mises, Rankine, Mohr-Coulomb & Drucker-Prager (de Souza Neto et al. 2008, Linero & Garzón-Alvarado 2010), as well as from criteria especially developed for concrete, such as the ones proposed by Ottosen (1977), Willam and Warnke (1974), Menetrey and Willam (1995), Hsieh and coworkers (1982), Hu and Schnobrich (1989), among others.

The flow rule defines the course of plastic strain rate in accordance with plastic potential variation G , in regards to tensile stress. If the chosen plastic potential is equal to yielding function, the flow rule will be denominated as associated. Contrarily, if G corresponds to a different function, a non-associated flow rule will be obtained. Generally the plastic potential function having non-associated flow rules are defined from classic failure criteria, such as developed by von Mises and Drucker-Prager.

Some specific plasticity constitutive models developed over the last decades for concrete are the following: The Han and Chen (1987) model, which employs the William and Warnke (1974) five parameters yielding function and a non-associated flow rule. Ohtami and Chen (1988) employed Chen and Chen (1975) yielding function and an associated flow rule. Oñate and coworkers modified Mohr-Coulomb criterion to be used as yielding function together with a non-associated flow rule. Fenestra and de Borst (1996) developed a model based on the yielding criterion by Drucker-Prager and non associative flow rule applicable on a composed material made up by a concrete matrix and steel bars. Imran and Pantazopoulou (Imran & Pantazopoulou 2001) employed Hsieh-Ting-Chen (Hsieh et al. 1982) four parameter criteria and a non-associated flow rule. The Grassl and others (2002) model employs the Menetrey and Willam (1995) yielding function, which is combined with a non-associated flow rule.

The overall objective of this research is to represent the behavior of simple concrete structures by means of a finite element method for non-linear problems, which non-associated constitutive model (de Souza Neto et al., 2008, Simó & Hughes 1998) employs the Hu and Schnobrich (1989) yielding function and the plastic potential function by von Mises.

2. Formulation of constitutive model

The elements of the proposed constitutive model, including yielding function, flow rule, hardening law and tangent constitutive equation are briefly described below.

Strain tensor ϵ and its rate $\dot{\epsilon}$ can be split into two parts: elastic strain ϵ^e and plastic strain ϵ^p , so that:

$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^e + \boldsymbol{\varepsilon}^p, \quad \dot{\boldsymbol{\varepsilon}} = \dot{\boldsymbol{\varepsilon}}^e + \dot{\boldsymbol{\varepsilon}}^p \quad (1)$$

It becomes a plastic potential function, as:

$$g(\boldsymbol{\sigma}, \bar{\sigma}) = G(\boldsymbol{\sigma}) - \bar{\sigma} = 0 \quad (2)$$

Plastic strain rate $\dot{\boldsymbol{\varepsilon}}^p$ is defined by the flow rule in terms of a potential plastic variation $g(\boldsymbol{\sigma}, \bar{\sigma})$ in regards to stress tensor, such as:

$$\dot{\boldsymbol{\varepsilon}}^p = \gamma \frac{\partial g}{\partial \boldsymbol{\sigma}} = \gamma \frac{\partial G}{\partial \boldsymbol{\sigma}} \quad (3)$$

Thus being $\gamma = \dot{\boldsymbol{\varepsilon}}^p$ the plastic multiplier or the equivalent plastic strain rate. Such multiplier is positive ($\gamma > 0$) due to the irreversible nature of plastic strain.

Yielding function $f(\boldsymbol{\sigma}, \bar{\varepsilon}^p)$ is established when a elastic stress status $\boldsymbol{\sigma}$ is associated with $\bar{\varepsilon}^p$ at an instant during loading process. In other words, an elastic loading or unloading status takes place if $f(\boldsymbol{\sigma}, \bar{\varepsilon}^p) < 0$ and a plastic loading status takes place when $f(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = 0$. Such function is given as the difference between two scale functions; the first $F(\boldsymbol{\sigma})$ is determined by the material yielding criterion in function of stress status and; the second $\bar{\sigma}(\bar{\varepsilon}^p)$ corresponds to an equivalent uniaxial stress, which depends on loading history and material hardening or softening law in terms of equivalent plastic strain, so that:

$$f(\boldsymbol{\sigma}, \bar{\varepsilon}^p) = F(\boldsymbol{\sigma}) - \bar{\sigma}(\bar{\varepsilon}^p) \leq 0 \quad (4)$$

Particularly, this model employs the yielding function defined by Hu and Schnobrich (Hu & Schnobrich 1989), from an experiment on simple concrete panels, for plain stress status. Such study establishes different behaviors for main biaxial tensile stress status ($\sigma_1 > 0, \sigma_2 > 0$), biaxial compressive stress ($\sigma_1 < 0, \sigma_2 < 0$) and tensile - compressive stress ($\sigma_1 > 0, \sigma_2 < 0$).

For a biaxial tensile status, that is to say if $\sigma_1 > 0$ and $\sigma_2 > 0$, the yielding function $f = F - \bar{\sigma} \leq 0$ is made up by functions:

$$F = c_1 \left(\frac{3}{2\sqrt{2}} \frac{(1+\alpha)}{\alpha} \tau_{oct} + \frac{3}{2} \frac{(1-\alpha)}{\alpha} \sigma_m \right), \quad \bar{\sigma} = \sigma_{ye} \quad (5)$$

$$c_1 = 1 - 0.4019 \left(\frac{\sigma_2}{\sigma_1} \right) + 0.008913 \left(\frac{\sigma_2}{\sigma_1} \right)^2, \quad \sigma_1 \geq \sigma_2$$

The average stress, the octahedral stress and the relation of tensile and compressive strengths correspond to:

$$\sigma_m = \frac{1}{3}(\sigma_1 + \sigma_2), \quad \tau_{oct} = \frac{\sqrt{2}}{3} \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2}, \quad \alpha = \frac{\sigma_{yt}}{\sigma_{ye}} \quad (6)$$

We can observe that the equivalent uniaxial stress is equal to the material maximum compressive stress σ_{yc} and, therefore, it does not evolve in regards to the equivalent plastic strain.

For a tensile-compressive stress status, where $\sigma_1 > 0$ and $\sigma_2 < 0$, the yielding function $f = F - \bar{\sigma}$ is equal to:

$$f = F - \bar{\sigma} = c_2 \left(\frac{3}{2\sqrt{2}} \frac{(1+\alpha)}{\alpha} \tau_{oct} + \frac{3}{2} \frac{(1-\alpha)}{\alpha} \sigma_m \right) - \bar{\sigma} \leq 0$$

$$c_2 = \begin{cases} 1 - 0.02886 \left(\frac{\sigma_2}{\sigma_1} \right) - 0.006657 \left(\frac{\sigma_2}{\sigma_1} \right)^2 - 0.0002443 \left(\frac{\sigma_2}{\sigma_1} \right)^3 & \text{si } \frac{\sigma_1}{\sigma_2} < -0.103 \\ 1 + 6.339 \left(\frac{\sigma_1}{\sigma_2} \right) + 68.82 \left(\frac{\sigma_1}{\sigma_2} \right)^2 + 183.8 \left(\frac{\sigma_1}{\sigma_2} \right)^3 & \text{si } \frac{\sigma_2}{\sigma_1} \geq -0.103 \end{cases} \quad (7)$$

The equivalent uniaxial stress $\bar{\sigma}$ is defined further on with the expressions (10) and (11), in terms of equivalent uniaxial strain.

For a biaxial compressive stress status, where the principal stresses are $\sigma_1 < 0$ and $\sigma_2 < 0$, the material softening is considered during the plastic stage, so that the yielding function $f = F - \bar{\sigma}$ will be equal to:

$$f = F - \bar{\sigma} = c_3 \left(\frac{3}{\sqrt{2}} \frac{(2\beta-1)}{\beta} \tau_{oct} + 3 \frac{(\beta-1)}{\beta} \sigma_m \right) - \bar{\sigma} \leq 0$$

$$c_3 = 1 - 0.05848 \left(\frac{\sigma_2}{\sigma_1} \right) - 0.05848 \left(\frac{\sigma_2}{\sigma_1} \right)^2 \quad \beta = 1.16 \quad \sigma_1 \geq \sigma_2 \quad (8)$$

The yielding function by Hu and Schnobrich determines the evolution of yielding surface during the inelastic material behavior, by means of a equivalent uniaxial non-linear function of the stress $\bar{\sigma}$ and the strain $\bar{\epsilon} = \bar{\epsilon}^e + \bar{\epsilon}^p$. By considering the initial elasticity modulus and the secant elasticity modulus as equals at $\bar{\sigma} = \sigma_{yc}$ the following expression for compressive equivalent uniaxial stress is obtained:

$$\bar{\sigma} = \frac{E_0 \bar{\epsilon}}{1 + (R-1) \left(\frac{\bar{\epsilon}}{\bar{\epsilon}_0} \right) - (2R-1) \left(\frac{\bar{\epsilon}}{\bar{\epsilon}_0} \right)^2 + R \left(\frac{\bar{\epsilon}}{\bar{\epsilon}_0} \right)^3} \quad (9)$$

$$R = \frac{(R_\sigma - 1)}{(R_\epsilon - 1)^2} - \frac{1}{R_\epsilon}, \quad R_\sigma = \frac{\sigma_{yc}}{\bar{\sigma}_f}, \quad R_\epsilon = \frac{\bar{\epsilon}_f}{\bar{\epsilon}_0}, \quad E_0 = \frac{\sigma_{yc}}{\bar{\epsilon}_0}$$

Being $\bar{\sigma}_f$ y $\bar{\epsilon}_f$ the stress and the strain at the end of softening branch of curve between the equivalent uniaxial stress and strain $\bar{\sigma} - \bar{\epsilon}$. Provided the difficulty of experimentally obtaining such material parameters, it is supposed that $R_\sigma = R_\epsilon = 4$ as proposed by Elwi and Murray (1979). The compression equivalent uniaxial stress expression is reduced to:

$$\bar{\sigma} = \Xi E_0 \bar{\varepsilon} \quad \text{donde} \quad \Xi = \left[1 - \frac{11}{12} \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}_0} \right) + \frac{10}{12} \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}_0} \right)^2 + \frac{1}{12} \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}_0} \right)^3 \right]^{-1} \quad (10)$$

As depicted in **Figure 1(a)**, the curve $\bar{\sigma}(\bar{\varepsilon})$ is not exactly linear before reaching the maximum compression stress σ_{yc} . However, the implemented model has considered an explicit linear relationship within this range, so that:

$$\bar{\sigma} = \begin{cases} 1 & \text{si } \bar{\varepsilon} \leq \bar{\varepsilon}_0 \\ \Xi E_0 \bar{\varepsilon} & \text{si } \bar{\varepsilon} > \bar{\varepsilon}_0 \end{cases} \quad (11)$$

The tangent elasticity modulus of the relationship between the stress and equivalent uniaxial strain corresponds to:

$$E_t = \frac{d\bar{\sigma}}{d\bar{\varepsilon}} = E_0 \left[12 - 10 \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}_0} \right)^2 - 2 \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}_0} \right)^3 \right] \left[1 - \frac{11}{12} \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}_0} \right) + \frac{10}{12} \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}_0} \right)^2 + \left(\frac{\bar{\varepsilon}}{\bar{\varepsilon}_0} \right)^3 \right]^{-1} \quad (12)$$

On the other hand a bilinear relationship is considered between the tensile equivalent uniaxial stress and strain, divided by the maximum tensile stress $\sigma_{yt} = \alpha \sigma_{yc}$ as indicated on **Figure 1(b)**. Hu and Schnobrich propose that biaxial tensile yielding surface will not evolve during the inelastic range; consequently the slope in the second interval of the curve will be zero, maintaining $\bar{\sigma} = \sigma_{yt}$.

The actual tensile behavior of concrete and the cohesive fracture models, among others (Bazant & Oh 1983, Graffe & Linero 2010); undergo a quick strength reduction after achieving the maximum tensile stress. However, the hypothesis in above paragraph is valid when it comes to represent stress status where compression prevails.

Figure 1. Relation between stress and equivalent uniaxial strain: a) compressive stress and b) tensile stress

The yielding function in the principal stress space $\sigma_1 - \sigma_2$ is represented by an initial yielding surface that holds stress status within the elastic range, which is progressively reduced upon loading evolution within the inelastic range, as indicated on **Figure 2**.

Figure 2. Evolution of yielding surface of the represented model for the principal stress space under plain stress condition

Plastic hardening – softening modulus under compression H is defined by the variation of equivalent uniaxial stress in regards to equivalent plastic strain, so:

$$H(\bar{\varepsilon}^p) = \frac{\partial \bar{\sigma}}{\partial \bar{\varepsilon}^p} \quad (13)$$

In accordance with strain additive decomposition and with the definition of uniaxial tangent elasticity modulus, we obtain:

$$\frac{d\bar{\varepsilon}}{d\bar{\sigma}} = \frac{d\bar{\varepsilon}^e}{d\bar{\sigma}} + \frac{d\bar{\varepsilon}^p}{d\bar{\sigma}} \rightarrow \frac{1}{E_t} = \frac{1}{E_0} + \frac{1}{H} \rightarrow H(\bar{\varepsilon}^p) = \frac{E_0 E_t(\bar{\varepsilon}^p)}{E_0 - E_t(\bar{\varepsilon}^p)} \quad (14)$$

Loading and unloading conditions established for a plastic loading status: if $\gamma > 0$ then $\gamma \dot{f} > 0$ for $\dot{f} = 0$, consequently $\dot{f} = 0$.

Based on the relation between tensile stress and elastic strain, the following secant constitutive equation can be defined:

$$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}^e = \mathbf{C} : (\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p) \quad (15)$$

Where $\mathbf{C}$ is the material elastic constitutive tensor

After differentiating the yielding function throughout time and conducting the corresponding replacements, the plastic multiplier and the tangent constitutive equation are obtained, as follows:

$$\gamma = \frac{\frac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbf{C}}{\frac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \frac{\partial G}{\partial \boldsymbol{\sigma}} + H} > 0 \quad (16)$$

$$\dot{\boldsymbol{\sigma}} = \mathbf{C}^{\text{tan}} : \dot{\boldsymbol{\varepsilon}} \quad \text{siendo} \quad \mathbf{C}^{\text{tan}} = \mathbf{C} - \frac{\mathbf{C} : \frac{\partial G}{\partial \boldsymbol{\sigma}} \otimes \frac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbf{C}}{\frac{\partial F}{\partial \boldsymbol{\sigma}} : \mathbf{C} : \frac{\partial G}{\partial \boldsymbol{\sigma}} - H} \quad \text{si } \gamma > 0 \quad \wedge \quad \mathbf{C}^{\text{tan}} = \mathbf{C} \quad \text{si } \gamma = 0 \quad (17)$$

Where $\mathbf{C}^{\text{tan}}$ and $\mathbf{C}$ are the elastic constitutive tensors and the tangent, respectively. It can be observed that tensor $\mathbf{C}^{\text{tan}}$ is a symmetrical tensor provided that the G function is equal or proportional to F function, which is called associated flow rule model.

The introduced model has a non associated flow rule, that it to say, the plastic potential function is different to the yielding function, consequently $g \neq f$ and $G \neq F$. The chosen potential function corresponds to a von Mises function under stress plain condition, described as follows:

$$g(\boldsymbol{\sigma}, \bar{\sigma}) = \frac{3}{\sqrt{2}} \tau_{oct} - \bar{\sigma} = 0 \quad , \quad G(\boldsymbol{\sigma}) = \frac{3}{\sqrt{2}} \tau_{oct} = \sqrt{\sigma_1^2 + \sigma_2^2 - \sigma_1 \sigma_2} \quad (18)$$

When employing a non associated flow rule, principles proposed by Drucker (Drucker 1950, Drucker 1951) are not fulfilled in regards to local material stability. However, such conditions are not strictly required (Mroz 1963) and they do allow us to obtain results derived from experimental tests.

As indicated by Equation (17), the tangent constitutive tensor $\mathbf{C}^{\text{tan}}$ depends, among others, on two second order tensors, which are the result of the partial derivative of a plastic potential function G and from a yielding function F in regards to tensile stress. From plastic potential function on Equation (18), the tensor components $\partial G / \partial \boldsymbol{\sigma}$ were calculated in function of principal stress σ_1, σ_2 . Based on the yielding function for stress states of stress states of biaxial tension, tension - compression and biaxial compression, expressed on equations (5), (7) and (8), respectively, the tensor components $\partial F / \partial \boldsymbol{\sigma}$ were obtained in terms of σ_1, σ_2 (Rodríguez-Fajardo 2011).

3. Implementation of numerical model on the finite elements method

This research incorporates the sub-routines of a new material constitutive model to the program code of on non linear analysis program by finite elements HYPLAS (de Souza Neto et al., 2008), by following an implicit integration scheme.

The implemented numeric solution considers a plain stress condition, infinitesimal strain and static loads.

Based on strain components and internal variables at an integration point of finite elements, a sub-routine of the constitutive model calculates the stress components and another obtains the tangent constitutive tensor coefficients on Voigt notation.

4. Model validation by means of numerical simulation on concrete panels test.

Kupfer and others (Kupfer et al., 1969) tested a set of simple concrete square panels of 0.20 m width and 0.05m thickness, subject to different distributed load status which is parallel to their plane, as indicated on **Figure 3**. Concrete has the following features: Young modulus $E_c = 20.97 \times 10^6 \text{ kN/m}^2$, Poisson relation $\nu = 0.20$, maximum compressive stress $\sigma_{yc} = 28.98 \times 10^3 \text{ kN/m}^2$ and a relation between maximum tensile and compressive stress $\alpha = \sigma_{yc}/\sigma_{yt} = 0.09$. Some of these tests were simulated by the proposed numerical model.

Figure 3. Simple concrete panels subject to: (a) biaxial tensile stress, (b) biaxial compressive stress and (c) tensile – compressive stress

Loads applied produce a homogeneous distribution of the stress field on the panel. Consequently, the problem was simulated with a mesh of a quadrilateral linear finite element of 4 nodes. The displacement in x direction on the left face and the displacement in y direction on the lower face are restrained, as indicated on Figure 4.

When changing pressure loads on the element faces, equal displacements are applied in each direction, in order to avoid convergence difficulties on the numeric solution when there is softening as structural response. Displacements are applied in the same direction and proportion as the loads endured by one face, with regards to the adjacent face. Supports are located on the nodes and they are allocated in such a way the element expansion or contraction is freely allowed.

Figure 4. Linear quadrilateral finite element mesh, contour conditions and applied actions

Displacements are gradually applied in small increments by simulating gradual load of laboratory tests.

The numerical simulation on different experimental tests was conducted by using the same mesh, only modifying the applied displacements. The relationship between displacement applied vertically on nodes 3 or 4, in regards to applied horizontal displacements on nodes 2 or 3, will determine the stress status in each test.

Table 1 indicates experimental tests that were simulated by using the proposed numeric model.

Table 1. Experimental tests that were simulated by using the proposed numeric model

Referencia/Reference	Estado de esfuerzo/Stress status	$\delta_y/\delta_x = \delta_1/\delta_2$
S-1	Compresión biaxial/Biaxial compression	-1.00 / -1.00
S-2	Compresión biaxial/ Biaxial compression	-1.00 / -0.52
S-3	Tracción - Compresión/Tensile - compression	-1.00 / -0.052
S-4	Tracción biaxial/ Biaxial tensile	+1.00 / +1.00

Figure 5(a) indicates the relation between regular stress and longitudinal strain in direction y , when equal displacements that generate compression in directions x and y are applied. Numerical results show a linear relation of approximately 80% of the maximum compressive stress. From this stress range, a non linear behavior takes place, which is divided into an ascendant branch up to $1.2\sigma_{yc}$ and other descendant branch that represent softening caused by strain. The numerical result approximately matches the experimental result developed by Kupfer and others (Kupfer et al., 1969).

Figure 5. Relation between regular stress and longitudinal strain in direction 1, for different stress status (a) biaxial compression $-1.00/-1.00$, (b) biaxial compression $-1.00/-0.52$, (c) tension – compression $-1.00/+0.052$ and (d) biaxial tension $+1.00/+1.00$.

Figure 5(b) shows the relation $\sigma_1/\sigma_{yc} - \epsilon_1$ for a panel subject to displacements that generate uniform compression in direction x, 0.52 times higher than the uniform compression in direction y. An initial linear behavior similar to the experimental one is observed, as well as a stiffer response from the numerical model in the non linear branch. However, the maximum stress is approximately equal to the experimental one.

Figure 5(c) presents the relation $\sigma_1/\sigma_{yc} - \epsilon_1$ for a panel subject to displacements that generate uniform compression in direction x, 0.052 times higher than uniform tensile stress in direction y. The initial linear relationship in the numerical model curve is different to the experimental behavior. The maximum stress obtained from the modeling and the experiment differs in 20% due to the fact that fracture phenomena are not taken into account, among other factors.

From numerical modeling of a panel subject to uniform biaxial tensile stress of the same magnitude, the following can be stated. The result obtained from the model is approximately the same as the experimental one, as shown on Figure 5(d). The constitutive model assumes a perfect plastic behavior after reaching the maximum tensile stress; that is why a structural response is obtained up to a higher strain level compared to the experimental result.

5. Application of the model on bending beams

A beam of length 3.00m and rectangular cross section of 0.25m base per 0.50m height are subject to a load on the right end. The material beam is represented by the introduced constitutive model and its mechanical properties are the following: Young module $E_c = 36 \times 10^6 \text{ kN/m}^2$, Poisson relation $\nu=0.19$, maximum compressive stress $\sigma_{yc} = 23950 \text{ kN/m}^2$, and relation between maximum tensile and compressive stress $\alpha = 0.09$.

The problem is simplified to a stress plane status by using a mesh of 24 linear quadrilateral finite elements, as shown on Figure 6.

Figure 6. Cantilever beam subject to point load. Finite elements mat

The load on the beam in end P evolves from an initial zero value, as indicated on Figure 7. First the load is applied in negative direction y exceeding the yielding strain at the higher moment sector (Top end fibers located next to the support). Afterwards load decreases until reversing way and to cause yielding strain once again now (for bottom fibers next to the support). Finally load decreases to zero.

Figure 7. cantilever beam subject to point load. Loading history

Figure 8. Cantilever beam subject to point load. Relationship between applied load point and vertical displacement on the right end

Figure 8 depicts the relationship between applied load point and vertical displacement on the right end. Highlighted dots in the curve from numbers 1 to 7 refer to representative loading steps. In the interval delimited by instants 1 and 2, the behavior of all beam material points is a linear elastic. Once instant 2 is exceeded, tensile yielding takes place on fibers close to the fixed support. In the interval 2 up to 3, load keeps increasing under a plastic loading process.

Afterwards, an elastic unload is applied on the interval delimited between dots 3 and 4. Such unload shows the same loading slope within an elastic range. It is observed that applied load is zero at instant 4; however there is a permanent displacement equal to 0.2mm at the beam end. Load increases in a reverse direction from instants 4 to 5, thus developing a reload within an elastic range. Then load is increased even more between dots 5 and 6, developing a tensile plastic load at the beam bottom fibers which are close to the fixed support. Load is elastically unloaded from dot 6 up to 7 until zero. We can observe a permanent displacement at the beam end at instant 7.

In spite of softening taking place, which indicates material behavior after the elastic tensile limit, the load-displacement curve shows a positive slope within the inelastic load stage. Above is caused because the beam area subject to compression maintains an elastic behavior due to factor α .

Figure 9 shows the distribution of normal stress component in direction x on the whole beam. There is a beam elastic structural response between steps 1 and 2, where stresses are symmetrical in regards to the neutral axis located on the beam centroidal axis. In this interval there is also a stress distribution, which is linearly increased as long as dots move away from the beam neutral axis.

In the load step 3, there is a tensile yielding strain on beam top fibers close to the fixed support. The stress on areas with plastic strain decreases progressively, due to material softening properties. In this loading step, the stress distribution on sections close to the fixed support is no linear and the neutral axis is displaced downwards to maintain the stress balance on each cross section. In spite of the increased applied load, the beam areas subject to a tensile stress maintain a steady constant value after achieving the elastic limit. Consequently, the neutral axis is displaced downwards, thus increasing the transversal section area subject to tensile stress on the upper side, which balances compressive stresses at the bottom side of such section. Compressive stresses rise as long as load is increased on the cantilever, because the elastic limit within this range is not exceeded.

Afterwards, when the end of a free load beam changes direction, the fibers subject to tensile stress correspond to the ones at the beam bottom side. At the load instant 5 showed by Figure 9(c), there is a change of stress distribution during plastic load process.

At instant 6 the same effect as described for instant 3 takes place. The tensile stress on the bottom side close to the fixed support decreases, as long as load is increased on the beam free end due to material softening. The beam neutral axis is now displaced towards the top side, because the regular tensile stress is reduced on the beam bottom side close to the fixed support. Therefore, a greater section area with tensile stresses is required on the bottom side in order to balance great compressive stresses taking place on the beam top side.

Figure 9. Cantilever beam subject to point load. Distribution of normal stress component in direction x for representative loading steps.

6. Conclusions

By considering an initial linear relationship between stress and equivalent uniaxial strain on Hu and Schnobrich model, the implementation of the proposed model is simplified without altering the adjustment of results when compared to the experimental data.

This research formulated and implemented an implicit integration algorithm from Hu and Schnobrich model, under the framework of a non linear analysis including finite elements, which the equilibrium conditions in each loading step are fulfill.

If increased displacements are applied instead of increasing loading by means of Newton – Raphson, method numerical convergence will be achieved with few iteration in problems where displacement increases and applied load decreases. It was also observed that by controlling load or displacement, by using the arc length method, a numerical convergence is achieved for the solution with little iterations.

The implemented constitutive model employs a non associated flow rule, which produce tangent constitutive tensors not to be symmetrical. At the same time the structure stiffness matrix is not symmetrical either. Above leads to a higher computational cost, however obtained results are quite close to the experimental ones in comparison to the results achieved by the associate flow rule model.

In accordance with results obtained by the implemented model, it is observed that the best adjustment is achieved within the range of biaxial compressive and biaxial tensile stress. For tensile – compressive range, the model has not the same concordance level that experimental results, since the model does not consider concrete phenomena such as fracture.

For biaxial tension, the selected model assumes, a perfect plasticity behavior. Above in order to avoid the solution numeric instability that takes place when stress quickly decreases as long as strain increases.

The proposed numerical model combines the mathematic principle of plasticity theory using an empirical approximation of concrete yielding function. Above leads to numerical simulation results that properly represent concrete mechanical behavior, mainly for biaxial compressive stress status, where phenomena such as fractures do not directly affect the behavior.

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